The Optimum Quantity of Capital and Debt*

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Abstract: In this paper we solve the dynamic optimal Ramsey taxation problem in a model with incomplete markets, where the government commits itself ex-ante to a time path of labor taxes, capital taxes and debt to maximize the discounted sum of agents' utility starting from today. Whereas the literature has been limited mainly to studying policies that maximize steady-state welfare only, we instead characterize the optimal policy along the full transition path.

We show theoretically that in the long run the capital stock satisfies the modified golden rule. More importantly, we prove that in contrast to complete markets economies, in incomplete markets economies the long run steady-state resulting from an infinite sequence of optimal policy choices is independent of initial conditions. This result is not only of theoretical interest but moreover, enables us to compute the long-run optimum independently from the transition path such that a quantitative analysis becomes tractable.

Quantitatively we find, robustly across various calibrations, that in the long run the government debt-to-GDP ratio is high, capital is taxed at a low rate and labor income at a high rate when compared to current U.S. values. Along the optimal transition to the steady state, labor taxes initially are lowered, financed through issuing more debt and taxing capital income heavily, before they are eventually increased to their steady-state level.

Keywords: Optimal Government Debt, Incomplete Markets, Capital Taxation, Dynamically Optimal Taxation

JEL: E62, H20, H60

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1 Introduction

What are the optimal levels of capital and government debt? Should capital be taxed and if yes, how much? What is the optimal extent of redistribution? We study these classic questions in a heterogenous agents, incomplete markets, Aiyagari (1995) economy. In this economy households are exposed to idiosyncratic income shocks but no aggregate risk. They face exogenous credit constraints and the only assets are physical capital and government debt. The Ramsey planner commits itself ex-ante to a path of linear labor and capital taxes and government debt to maximize agents' discounted present value of lifetime utility.

We prove two main theoretical findings on optimal policies. First, we show that it is optimal to equalize the pre-tax return on capital and the rate of time preference in the long-run, i.e. the capital stock satisfies the modified golden rule. Our second theoretical result shows that the long-run steady-state allocations and policies are independent of initial conditions. In particular, the long-run level of government debt is uniquely determined and does not depend on the initial value of debt or capital. Similarly, steady-state tax rates on capital and labor are unique and independent of initial conditions.

The result that the long-run steady state is independent of initial conditions is not only of theoretical interest but also renders tractable a quantitative analysis of the dynamic optimal taxation problem. Whereas the literature has focused mainly on characterizing the steady state which maximizes welfare, we develop a new computational algorithm that allows us to maximize welfare in the initial period by choosing the optimal path of taxes and debt.

A comparison with the optimal Ramsey taxation results in representative agents, complete markets economies without aggregate risk as in Lucas (1990), and Chari and Kehoe (1999) helps to elucidate our theoretical findings. As is well known the steady-state Ramsey planner solution depends on initial conditions, such as the initial government debt level, in this complete markets environment. The intuition for this result is straightforward. As in Barro (1979) the planner aims to smooth distortions over time using government debt. In the absence of any exogenous fluctuations it is optimal (perhaps after a few early periods) to keep government debt and labor taxes constant over time. This policy provides higher welfare than a deviating policy where, for example, labor taxes and distortions are lowered initially, more debt is issued to finance this tax cut, and then eventually labor taxes and distortions are increased to cover the higher interest rate burden on government debt. This alternative policy would reduce welfare since the gain of lower distortions in the beginning is outweighed by the loss of higher distortions later on, since distortions are "convex" as in

Barro (1979).

If markets are incomplete though, this reasoning is only one part of the story (Heathcote, 2005). Lowering taxes today still means higher debt (as in the complete markets case) but now more debt has a welfare-enhancing element, as it improves households' ability to smooth consumption in response to income shocks. The costs of having higher debt - higher future taxes -remain if markets are incomplete, but there is now an additional benefit: better consumption smoothing. As a result the planner lowers taxes initially in light of two benefits - lower distortions today and higher debt (more liquidity) - and still just one cost (higher distortions tomorrow). Of course, there are limits to how high debt can become; eventually, future distortions get too big and outweigh the initial lower distortions and the benefits of higher liquidity. The optimal level of government debt is determined as equalizing the benefits and costs at the margin.

A conclusion common to both complete and incomplete markets is that the long-run capital stock satisfies the modified golden rule (see also Aiyagari, 1995).[1] In a representative agent economy distributional concerns are absent and investment efficiently transfers resources across time. If markets are incomplete distributional concerns are present, but we show that these do not interfere with efficiency in investment, reminiscent of the production efficiency result in Diamond and Mirrlees (1971). Aiyagari (1995) shows that this finding implies a positive capital income tax rate. One interpretation here is that this tax corrects households' overaccumulation of capital due to a precautionary savings motive. However, we show this interpretation to be inaccurate and that instead, the planner issues as much debt as is necessary to enhance consumption smoothing such that capital demand satisfies the modified golden rule. The capital income tax rate is positive, as Aiyagari's (1995) arguments are valid, and is such that the private sector is willing to absorb the optimal levels of both capital and government debt. Thus there is no need to implement a higher than efficient capital stock in order to achieve better consumption smoothing, simply because debt can be used instead to prevent the overaccumulation of capital. A higher capital stock could also be used to increase wages, which would benefit those depending primarily on labor income but we show that the planner could use labor taxes to increase the after-tax wage instead. On the other hand, if either of the instruments, issuing debt or taxing labor, is unavailable to the planner, then the capital stock will not satisfy the modified golden rule.

Our result showing independence of initial conditions allows for a quantitative analysis of

[1]Park (2014), however, shows that this result does not extend to her limited commitment economy.

the optimal dynamic taxation problem. Whereas the literature has focused on maximizing steady-state welfare, our task is to characterize the optimal policy along the full transition path. In particular our characterization has to take into account that the optimal policy, at each point in time during the transition, depends on the full transition path of capital, debt and tax rates.

Computing the path of tax rates, government debt and transfers that maximizes welfare in the initial period is a huge computational challenge. Several hundreds or thousands of variables must be chosen in a highly nonlinear optimization problem. However, our result that the optimal long-run policy is independent of initial conditions turns this unwieldly optimization problem into a manageable one. From a computational point of view, independence of initial conditions means that we know the optimal long-run policies and allocations without having to compute the transition. We know the initial conditions (economy calibrated to the US economy) and we know the terminal condition, the optimal long-run steady state characterized above. The (still daunting) computational problem is then to find the policy path that satisfies all necessary first-order conditions along the transition and at the same time the initial and terminal conditions. This is a challenge as it involves solving hundreds or thousands of nonlinear equations but it is significantly easier, and tractable as opposed to the original problem, which was to find the optimal transition and the optimal terminal point at the same time. Further, given the large number of variables involved in the original problem, there is no way to check whether a candidate solution is a global maximum. This is not a concern in our approach.

In the optimal steady state we find that capital taxes are always significantly positive in contrast to complete markets (see the seminal contributions of Chamley, 1986; Judd, 1985) although, for all calibrations, relatively low compared to most developed economies. In our benchmark calibration, aimed at resembling the high income inequality in the U.S. and with a Frisch elasticity of labor supply equal to one, the long-run taxes on capital and labor are around 11 and 77 percent. The optimal long-run level of government debt equals 4 times GDP.

Our finding that government debt is high, capital is taxed at a low rate, and labor income is taxed at a high rate when compared to current U.S. values, is robust across various different alternative calibrations, although the precise numbers do depend on the details of the calibration. Indeed, we reach the same conclusion for a low and a high Frisch elasticity of labor supply, for a low or high income elasticity of labor supply, for low and high income inequalities and in a model with permanent income differences.

The high debt levels we find follow also from our assumption that the government always honors its debt, so that elements such as a default premium, not present in our model, do not restrict how much debt can be issued. Instead distortionary taxes is the only element keeping debt from becoming infinitely large and thus maximizing the liquidity services. One conclusion from our result is that tax distortions alone restrict government debt to much higher levels than observed in developed countries[2].

Knowing the optimal path of policies allows us to compute the welfare gains of switching to the optimal policy and to better understand the properties of the optimal steady-state policies, as these are linked tightly to the policies chosen during the transition. The optimal transition is characterized by an initial period of high capital income taxation and low labor taxation. While the high initial capital tax rates are well known from complete markets and are a result of initially inelastically supplied capital, the low initial taxation (subsidization) of labor income is new to the incomplete markets environment. As a result labor market distortions are low initially and government debt accumulates. Eventually labor taxes are increased to pay the interest rates on debt which converges to its high steady-state level. The transition path also explains why the planner chooses a policy such that welfare in the terminal optimal steady state is lower than in the initial steady state. Since it is evaulated in the initial period welfare is an accumulation of the welfare gains and losses along the full transition path to the new steady state. Welfare is enhanced as the welfare gains of low labor taxation in the beginning of the transition outweigh the (highly discounted) welfare losses in the terminal steady state.

Although most of the literature either maximizes steady-state welfare or, when considering transitions, assumes fixed tax rates throughout the transition, a few papers deviate from these restrictive assumptions. For example, Dyrda and Pedroni (2015) also compute the optimal transition path in an incomplete markets economy, but use a quite different approach. In particular they do not characterize the optimal steady-state policies first, before computing the transition, but compute both jointly. Their findings for the optimal steady-state policy differ from ours: capital income in the long-run is taxed at a high rate whereas labor income is taxed at a low rate only, and government debt is negative. Complemetary to this paper, Le Grand and Ragot (2017) derive the optimal policy response to aggregate shocks. They find capital income taxes to be more volatile than labor taxes and government

[2]Holter et al. (2015) obtains similar results in an OLG economy.

debt to be countercyclical and mean-reverting.

Aiyagari and McGrattan (1998) study the optimal level of debt in an incomplete markets model but under the alternative assumption that the planner maximizes the utility at the steady state instead of ex-ante welfare. They find that the optimal level of debt is two-thirds of GDP, in line with the current US level. Much of the follow-up work in this literature also maximizes the steady-state welfare. For example, Röhrs and Winter (2014) find that if inequality is large, the optimal level of debt that maximizes the steady-state welfare is even lower and should be negative, -0.8. One reason why the optimal level of debt is low or even negative when steady-state welfare is maximized is that this optimality criterion ignores the welfare loss of reducing debt along the transition path to a low-debt steady state.

Another related paper, which maximizes steady-state welfare, is Conesa et al. (2009). They solve for optimal taxes on capital and labor (but not debt) in an OLG economy with incomplete markets and idiosyncratic income risk. They find a relatively high capital tax of 36%. Domeij and Heathcote (2004) are among the the first papers to look at the welfare impacts of taxes during transitions. They, however, study transitions after a once and forever lasting tax change, whereas this paper studies a transition with different taxes at every point in time. Kindermann and Krueger (2015) focus at how to tax individuals on the top of the income distribution. They solve for the once and forever change in the top marginal tax rate which maximizes period 1 welfare, and find that the optimal top marginal tax rate should be 90%.

In a series of papers Bhandari et al. (2015, 2016a,b, 2017) also consider optimal taxation in incomplete market models, building on the work of Aiyagari et al. (2002) who were the first to investigate the Ramsey policy in a Lucas and Stokey (1983) economy with incomplete markets (and aggregate risk). A key difference is that we follow Aiyagari (1995) and impose tight exogenous credit constraints, which is necessary for matching the joint distribution of earnings, consumption and wealth observed in the data and for generating a realistic distribution of marginal propensities to consume.[3] These credit constraints make the computational problem significantly more complicated, since a fraction of households is not operating on their consumption Euler equation, preventing us from using an easy backward shooting approach iterated on the Euler equation.

Tight credit constraints also seem to thwart a characterization of optimal policy through

[3]Tight credit constraints are also an important difference between this paper and Gottardi et al. (2015) as the latter paper assumes credit constraints to be non-binding.

sufficient statistics (Piketty and Saez, 2013) since they induce different policies and different distributions of assets, labor income and consumption in the short run, during the transition and in the long run to be optimal. Indeed we show that tight credit constraints and precautionary savings demand imply that it is optimal to increase the level of government debt and lower labor taxes initially and increase them in the long run. This induces a very different distribution of consumption, income and wealth in the long run from what is currently observed in the U.S. A sufficient statistics approach, however, is necessarily based on the observable inequality measures for the U.S. while optimal policy in the long run depends on the corresponding long-run statistics and their optimal evolution during the transition. Policy conclusions based on two very different statistics are likely to differ greatly. Furthermore it seems infeasible to solve the fixpoint problem - different policies lead to different wealth and income distribution, which renders different policies optimal and so on - within the sufficient statistic framework. Our results show that these considerations are not only a theoretical possibility but are key in determining the full transition path of optimal policies.[4]

It is also the presence of credit constraints that generates a large demand for precautionary savings and thus potentially a positive capital income tax rate. The reason why nevertheless we do not find high capital income tax rates is the large amount of debt, which allows households to smooth consumption quite well but requires an after-tax interest rate close to the rate of time preference. For a higher capital income tax rate and thus a lower pre-tax interest rate, the private sector would not be willing to absorb the capital stock and the large stock of debt. The planner finds it welfare-maximizing to reduce inequality through more debt and low capital income tax rates, instead of through low debt and high capital income taxes. Both a high level of debt and high capital tax rates are not possible since the asset market would not clear.

The paper is organized as follows. Section 2 presents our incomplete markets model and the Ramsey taxation problem. We provide our theoretical results in Section 3 before we move to the quantitative analysis. Section 4 shows optimal policy in the steady state and the optimal transition path is presented in Section 5. Section 6 concludes.

[4]One of the difficulties that arises if, for example, one is interested in finding the optimal capital income tax rate is that this requires specifying how the revenue from this tax is used: to lower the tax on labor, to pay higher transfers or to reduce government debt. This choice is not arbitrary but has to be optimal requiring a full account of its effects on the full transition path, which will in turn affect the optimal capital income tax rate, etc.

2 The Model

In this section, we present the incomplete markets model with heterogenous agents and uninsurable idiosyncratic labor productivity shocks. The setup is similar to Aiyagari (1995), except that our utility function is more general, which allows for income effect on labor supply and government spending is exogenous.

2.1 The Environment

Time is discrete and infinite, denoted by $t \in \{0, 1, 2, ...\}$. There is a continuum of ex ante identical households, a representative firm and a government.

Endowment and Technology A household supplies labor $n_t \in [0, 1]$ in period t. She faces an idiosyncratic labor productivity shock $e_t \in E$, which follows a Markov process and is i.i.d. across households. She has access to an incomplete market and can only hold a non-state contingent one-period bond $a_t \in A$, subject to a constraint $a_t \geq -\underline{a}$.

A representative competitive firm produces final goods using capital K_t and labor N_t using the neoclassical constant-returns-to scale production function $F(K, N)$ which satisfies the standard conditions.[5] Capital depreciates at rate δ.

The government is a Ramsey planner with full commitment. It collects linear capital income tax at the rate τ_{kt} and linear labor income tax at the rate τ_{nt}. It issues government debt B_t to finance lump-sum transfer T_t and government expenditure G_t.

Preferences The instantaneous utility of a household is $u(c_t, n_t) = \frac{c^{1-\sigma}}{1-\sigma} - \chi \frac{n^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}}$. Her lifetime utility is the expected discounted sum of utilities $\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$. This utility function allows for an income effect on labor supply, namely, the labor supply decision of a household reacts not only to the wage, but also to consumption and thus the level of assets. In the literature, to simplify the analysis, income effects on labor supply usually are shut down either by using, for example, GHH preferences or by allowing for home production.

Markets There are competitive markets for labor, capital, final goods, and bonds.

[5]The production function is assumed to be twice contiunuously differentiable, strictly increasing and concave in each argument and satisfying the standard Inada conditions: $\lim_{K \to 0} F_K = \infty, \lim_{K \to \infty} F_K = 0$ and $\lim_{N \to 0} F_N = \infty$.

2.2 Competitive Equilibrium

Firm The optimality conditions for the firm imply that in each period, the interest rate and the wage are equal to the marginal return of capital and the marginal return of labor respectively,

$$\begin{aligned} r_t &= F_K(K_t, N_t) - \delta, \\ w_t &= F_N(K_t, N_t). \end{aligned}$$

Government The government collects linear taxes on capital income and labor income. Denote the after-tax capital return and wage as $\bar{r}_t$ and $\bar{w}_t$, so that $\bar{r}_t = (1 - \tau_{kt}) r_t$ and $\bar{w}_t = (1 - \tau_{nt}) w_t$. The government's inter-temporal budget constraint is

$$G_t + (1 + \bar{r}_t) B_t + T_t \leq \tau_{kt} r_t A_t + \tau_{nt} w_t N_t + B_{t+1}, \tag{1}$$

where $A_t = K_t + B_t$ is the total amount of assets, the sum of physical capital and government debt. Standard arguments using the constant-return-to-scale assumption lead to the following equivalent resource constraint:

$$G_t + (1 + \bar{r}_t) B_t + \bar{r}_t K_t + \bar{w}_t N_t + T_t \leq F(K_t, N_t) - \delta K_t + B_{t+1}. \tag{2}$$

Households Starting from period 0 with asset a_0 and productivity e_0, a household solves the following problem

$$V_0(a_0, e_0) = \max_{\{a_{t+1}, c_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{n_t^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}} \right),$$

subject to

$$\begin{aligned} c_t + a_{t+1} &\leq a_t (1 + \bar{r}_t) + \bar{w}_t e_t n_t + T_t, && (3) \\ a_{t+1} &\geq -\underline{a}, && (4) \end{aligned}$$

where $V_0(a_0, e_0)$ represents the lifetime utility of a household with initial state (a_0, e_0). The optimality condition of n_t is

$$u_c(c_t, n_t)\, e_t \bar{w}_t + u_n(c_t, n_t) = 0,$$

which implies that labor supply

$$n_t = \left(\chi^{-1} e_t \bar{w}_t c_t^{-\sigma}\right)^{\phi}, \tag{5}$$

and after tax labor income

$$y_t = e_t n_t \bar{w}_t = (e_t \bar{w}_t)^{1+\phi} \left(\chi^{-1} c_t^{-\sigma}\right)^{\phi}. \tag{6}$$

In the rest of the paper, we can therefore treat n_t and y_t as known functions of $\bar{w}_t$ and c_t, reducing the number of choice variables. The optimality condition for a_{t+1} and the borrowing constraint imply the necessary conditions:

$$u_c(c_t, n_t) \geq \beta(1 + \bar{r}_{t+1})\, \mathbb{E}_t u_c(c_{t+1}, n_{t+1}), \tag{7}$$

$$(a_{t+1} + \underline{a})\left(u_c(c_t, n_t) - \beta(1 + \bar{r}_{t+1})\, \mathbb{E}_t u_c(c_{t+1}, n_{t+1})\right) = 0. \tag{8}$$

Equation (7) is the standard Euler equation, and equation (8) is the Kuhn-Tucker condition for the borrowing constraint.

Equilibrium The distribution of households with productivity e_t and asset a_t in period t is denoted by μ_t, a measure on $S = E \times A$. The asset market clearing conditions for assets, labor and capital are,

$$A_t = \int_S a_t d\mu_t, \tag{9}$$

$$N_t = \int_S e_t n_t d\mu_t, \tag{10}$$

$$K_t = A_t - B_t. \tag{11}$$

A sequence of prices and allocations and policies $\{\bar{r}_t, \bar{w}_t, T_t, B_{t+1}, K_{t+1}, a_{t+1}, c_t\}_{t=0}^{\infty}$ is a competitive equilibrium given initial conditions (B_0, K_0, μ_0) if

1. Households maximize utility (taking prices and policies as given).

2. Firms maximize profits (taking prices and policies as given).

3. Market clearing conditions (9), (10) and (11) hold.

2.3 The Optimal Taxation Problem

The Ramsey planner maximizes the sum of lifetime utilities of all households, by choosing time paths for $\bar{r}_t, \bar{w}_t$ and B_t consistent with equilibrium conditions described above. These are the instruments considered in the representative agent literature, which excludes lump-sum taxation since otherwise the planner would not have to use distortionary taxation. In our heterogeneous agents, incomplete markets model this restrictive assumption is not necessary. We also report results when choosing a path for transfers T_t as an additional instrument for the planner. We do not restrict the sign of T_t so that both transfers, $T_t > 0$, and lump-sum taxes, $T_t < 0$, are feasible. As explained, choosing the full time path distinguishes this paper from many other studies of optimal taxation in the literature, which e.g. maximize steady-state welfare. The Ramsey problem is

$$\max_{\{\bar{r}_t,\bar{w}_t,B_{t+1},T_t,a_{t+1},c_t\}} \int V_0(a_0, e_0) d\mu_0$$

subject to the resource constraint (2), households' budget constraints (3), households consumption Euler equation (7), and the credit constraint (8). The other unknowns, including $n_t, r_t, w_t, K_t, A_t, N_t$ can all be expressed as functions of the choice variables in the Ramsey problem, using the equations described in subsection 2.2.

One way to solve this problem would be to extend the primal approach used in complete markets models and to use first-order conditions to substitute for prices. We take a different route and we are the first to apply a Lagrangian maximization approach to study optimal taxation in an Aiyagari incomplete markets economy. We assign present value Lagrangian multipliers γ_t, θ_{t+1} and η_{t+1} to constraints (2), (7) and (8), respectively. The Lagrangian can be written (see Appendix I) as

$$\begin{aligned} \mathcal{L} \; = \; & \int \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \\ & \left(u\left(c_t, n_t\right) + u_c\left(c_t, n_t\right)\left(\left(\eta_t\left(a_t - \underline{a}\right) - \theta_t\right)\left(1 + \bar{r}_t\right) - \left(\eta_{t+1}\left(a_{t+1} - \underline{a}\right) - \theta_{t+1}\right)\right)\right) \end{aligned}$$

$$+\gamma_t\Big(F(K_t,N_t)-\delta K_t+B_{t+1}-G_t-T_t-(1+\bar{r}_t)B_t-\bar{r}_tK_t-\bar{w}_tN_t\Big)\Bigg\}d\mu_0. \quad (12)$$

To simplify the notation, we define $\lambda_{t+1}=\eta_{t+1}(a_{t+1}+\underline{a})-\theta_{t+1}$. We derive FOCs from the Lagrangian in Appendix I and show that the interior solution of the Ramsey problem satisfies the following conditions:

$$\begin{aligned}
\lambda_{t+1}: &\quad u_c(c_t)\geq\beta(1+\bar{r}_{t+1})\mathbb{E}_t[u_c(c_{t+1})] \\
&\quad \text{with equality if } a_{t+1}>-\underline{a}, && (13)\\
a_{t+1}: &\quad u_c(c_t)+\tfrac{\partial c_t}{\partial a_{t+1}}u_{cc}(c_t)(\lambda_t(1+\bar{r}_t)-\lambda_{t+1}) \\
&\quad =\beta\mathbb{E}_t\Big[(1+\bar{r}_{t+1})u_c(c_t)+\tfrac{\partial c_{t+1}}{\partial a_{t+1}}u_{cc}(c_{t+1})(\lambda_{t+1}(1+\bar{r}_{t+1})-\lambda_{t+2})\Big] \\
&\quad +\beta\gamma_{t+1}(F_K(K_{t+1},N_{t+1})-\delta-\bar{r}_{t+1}) \\
&\quad \text{if } a_{t+1}>-\underline{a};\ \text{otherwise } \lambda_{t+1}=0, && (14)\\
B_{t+1}: &\quad \gamma_t=\beta(1+F_K(K_{t+1},N_{t+1})-\delta)\gamma_{t+1}, && (15)\\
\bar{r}_t: &\quad \gamma_tA_t=\gamma_t(F_N(K_t,N_t)-\bar{w}_t)\tfrac{\partial N_t}{\partial\bar{r}_t} \\
&\quad +\mathbb{E}_t[u_c(c_t)\lambda_t+a_t(u_c(c_t)+u_{cc}(c_t)(\lambda_t(1+\bar{r}_t)-\lambda_{t+1}))], && (16)\\
\bar{w}_t: &\quad \gamma_tN_t=\gamma_t(F_N(K_t,N_t)-\bar{w}_t)\tfrac{\partial N_t}{\partial\bar{w}_t} \\
&\quad +\mathbb{E}_t[e_tn_t(u_c(c_t)+u_{cc}(c_t)(\lambda_t(1+\bar{r}_t)-\lambda_{t+1}))]. && (17)
\end{aligned}$$

$\frac{\partial c_t}{\partial a_{t+1}}$, $\frac{\partial c_{t+1}}{\partial a_{t+1}}$, etc. are known functions of control variables. The explicit expressions of these functions are shown in the Appendix. We simplify the notation of marginal utility of consumption as $u_c(c_t)$ instead of $u_c(c_t,n_t)$, given that the utility function is separable in consumption and labor. Note that as in Marcet and Marimon (2011), we expand the state space of the problem to include Lagrange multipliers of the dynamic implementability constraints. Since there is a continuum of heterogeneous households, the relevant state variable for the Ramsey planner is the joint distribution of these multipliers.[6] If transfers T_t are a choice

[6]Marcet and Marimon (2011) construct the recursive Lagrangian by "dualizing" the dynamic incentive constraints period by period, assuming that the solution to the primal problem is a saddle-point of the corresponding Lagrangian. An earlier draft of their paper (1994) features the Ramsey problem under complete markets as an example whose formulation looks very similar to this model.

variable for the planner we obtain an additional FOC,

$$T_t: \quad \gamma_t = \mathbb{E}_t \left[u_c(c_t) + \frac{\partial c_t}{\partial T_t} u_{cc}(c_t) \left(\lambda_t (1 + \bar{r}_t) - \lambda_{t+1} \right) \right] + \gamma_t \left(F_N(K_t, N_t) - \bar{w}_t \right) \frac{\partial N_t}{\partial T_t}. \tag{18}$$

3 Analytical Results

A key first step in the quantitative analysis is to compute the optimal policy in the long run. The second step is to use the optimal long-run policy as a terminal condition when computing the optimal policy during the transition path. We therefore make the standard assumption that the optimal long-run policy is stationary, an assumption we maintain throughout the paper:

Assumption 1. *For each set of initial conditions (B_0, K_0, μ_0), the economy (including policy and all other variables) converges to a unique steady state.*

Note that the above does not assume our main result on the independence of initial conditions. Instead it assumes that for each set of initial conditions (B_0, K_0, μ_0), there is a unique solution to the maximization problem of the Ramsey planner. Note that this assumption holds also in representative agent economies, where given the initial level of debt B_0 and capital K_0 the steady state is unique. But at the same time, the steady state depends on the initial debt level that is, different steady states can be reached for different initial conditions. In contrast, we show independence of initial conditions in our incomplete markets economy. The same steady state is reached independent of where the economy started.

Whereas uniqueness is a generic property of maximization problems (it just rules out more than one global maximum),[7] the second assumption that the optimal solution converges to a steady state is standard and essential for tractability in incomplete market models. Aiyagari (1995) therefore assumes that government expenditures are endogenous and constant in a steady state, implying finite values for the associated steady-state multipliers. It would be straightforward to follow Aiyagari (1995) and to incorporate endogenous government expenditures into our analysis without changing our conclusions. Chen et al. (2017) suggest that this assumption might be unnecessary, as finite values of the multipliers can be established if government expenditures are exogenous, confirming the claim in footnote 15 in Aiyagari

[7]See Aiyagari (1994b) for a proof that a solution to the optimal taxation problem exists.

(1995). Straub and Werning (2015) show that in a different model, the capitalist-worker model of Judd (1985) without government debt, the optimal solution does not converge to a steady state if the intertemporal elasticity of substitution is below one (and the welfare weight on capitalists is zero). For these parameter values, Proposition 2 in Straub and Werning (2015) shows that no interior steady state exists, implying that the assumption of convergence to an interior steady state is invalid. Specifically, they show that the multiplier on the first-order savings decision cannot converge to an interior steady-state value. In contrast, we prove the existence of steady-state Lagrange multipliers on households savings decisions in our incomplete markets taxation environment. Furthermore, issues relating to non-existence of steady-states do not arise in our numerical applications because we are always able to find a solution to the FOCs characterizing the steady state.[8] Whereas these findings show that existence of a steady state is not an issue in our incomplete markets model, little is known about whether the optimal solution converges to this steady state for arbitrary initial conditions (see footnote 14 in Aiyagari (1995)). That is why we impose the standard Assumption 1.

3.1 Steady State

This assumption on the stationarity of the optimal long-run policy means that we can replace all variables in the above FOC with their steady-state values. Then the optimal stationary policy is a solution to:

$$u_c(c) \geq \beta(1+\bar{r}')\,\mathbb{E}\left[u_c(c')\,|e\right] \text{ with equality if } a' > -\underline{a}, \tag{19}$$

$$u_c(c) + \frac{\partial c}{\partial a'}u_{cc}(c)\left[\lambda(1+\bar{r}) - \lambda'\right] = \beta\mathbb{E}\left[(1+\bar{r})\,u_c(c') + \frac{\partial c'}{\partial a'}u_{cc}(c')\left(\lambda'(1+\bar{r}) - \lambda''\right)\right]$$
$$+\beta\gamma\left(F_K(K,N) - \delta - \bar{r}\right) \text{ if } a' > -\underline{a}, \text{ otherwise } \lambda' = 0, \tag{20}$$

$$1 = \beta\left(1 + F_K(K,N) - \delta\right), \tag{21}$$

$$\gamma A = \gamma\left(F_N(K,N) - \bar{w}\right)\frac{\partial N}{\partial \bar{r}} + \mathbb{E}\left[u_c(c)\,\lambda\mu(ds,e) + a u_c(c) + \frac{\partial c}{\partial \bar{r}}u_{cc}(c)\left(\lambda(1+\bar{r}) - \lambda'\right)\right],$$

[8] Straub and Werning (2015) also consider the representative agent Ramsey taxation problem in Chamley (1986) and find that an exogenous upper bound on capital taxes can be binding forever if the initial level of government debt is close enough to the peak of a "Laffer curve". Again these issues seem not to arise in our incomplete markets model. We also impose an upper bound on capital taxation but find it to be binding only for the first period. Instead the planner finds it optimal to lower labor taxes and issue more bonds, which requires a sufficiently high after-tax return on assets if households are to be willing to absorb the additional debt.

$$\tag{22}$$

$$\gamma N = \gamma \left(F_N\left(K,N\right) - \bar{w}\right) \frac{\partial N}{\partial \bar{w}} + \mathbb{E}\left[enu_c\left(c\right) + \frac{\partial c}{\partial \bar{w}} u_{cc}\left(c\right)\left(\lambda\left(1+\bar{r}\right) - \lambda'\right)\right]. \tag{23}$$

again with the additional condition

$$\gamma = \mathbb{E}\left[u_c\left(c\right) + \frac{\partial c}{\partial T} u_{cc}\left(c\right)\left(\lambda\left(1+\bar{r}\right) - \lambda'\right)\right] + \gamma\left(F_N\left(K,N\right) - \bar{w}\right)\frac{\partial N}{\partial T}, \tag{24}$$

if transfers T_t are a choice variable of the planner.

3.2 Optimal Long-run Level of Capital

While most of our results naturally are based on numerical simulations, we can still analytically derive the optimal level of capital in the long run. A key property of the steady state is that the capital level satisfies the modified golden rule (see also Aiyagari (1995)). Equation (21) implies:

Theorem 1. *The steady-state capital stock satisfies the modified golden rule,*

$$\beta(1 + F_K\left(K,N\right) - \delta) = 1. \tag{25}$$

The modified golden rule states that it is optimal to equalize the return on capital and the rate of time preference, that is resources are efficiently allocated across time. This result is familiar from representative agent economies where distributional concerns are by assumption absent. Theorem 1 shows that we obtain the same efficiency result in our incomplete market economy where redistribution might induce a deviation from production efficiency, reminiscent of the production efficiency result in Diamond and Mirrlees (1971).

As is well known agents engage in precautionary savings to smooth consumption in response to idiosyncratic income fluctuations, and smoothing is more effective the more assets are available. The planner does not issue more capital to increase the availability of assets, but instead issues more government debt. This measure has the advantage that debt can be used as well as capital for consumption smoothing, but does not interfere with production efficiency. This reasoning is reflected in the absence of a "precautionary savings" term in the FOC determining the optimal level of capital.

A higher than efficient capital stock could also be used to increase wages, benefiting those whose consumption is financed primarily from labor and not asset income, as is the case in Dávila et al. (2012). In our Ramsey taxation problem the planner can increase the capital stock too but only by lowering capital income taxes. The planner can also use labor taxes to change the after-tax labor income.[9] We show that the planner uses labor taxes to modify the after-tax wage and not a higher capital stock, as is again reflected in the absence of a "wage" term in the FOC determining the optimal level of capital.[10]

These arguments establish, moreover, that the availability of government debt and labor taxes is necessary for theorem 1 to be valid. Without these instruments the modified golden rule does not hold. If labor taxes are unavailable, the planner needs to take into account that a higher capital stock leads to higher wages; and if government debt is unavailable, the planner needs to take into account that a higher capital stock improves consumption smoothing.

3.3 Optimal Long-run Level of Debt

As is well known, the steady state Ramsey planner solution depends on initial conditions, i.e. the initial government debt level, when markets are complete (see e.g. Lucas (1990) and Chari and Kehoe (1999)). The next theorem shows that this result is overturned if markets are incomplete.

Theorem 2. *The long-run values of government debt, of the labor income tax rate, and of the capital income tax rate are generically independent of the initial level of government bonds (and the initial capital stock).*

To better understand this result, it is important to recognize that the key difference between complete and incomplete market models is that households face credit constraints

[9]Dávila et al. (2012) study a different problem, the constrained efficient allocation in a model with exogenous labor, where the planner also maximizes the discounted present value of lifetime utility but decides how much each individual has to save without the need to implement those decisions through a properly designed tax scheme. They find, using a calibration similar to ours, that the optimal level of capital is much higher than the current U.S. level as the rich have to save more such that aggregate capital and thus wages increase.

[10]Lowering debt while keeping the total amount of households' assets constant increases capital but lowers the marginal product of capital (MPK). For a fixed after-tax interest rate $\bar{r}$ (which is necessary to keep total assets $K + B$ constant), a lower MPK is equivalent to lower capital income taxes.

in the incomplete markets world, and do not in the complete markets world. If markets are complete and thus lack credit constraints, the optimal steady state is linked to the initial steady state through the optimality conditions along the transition path. The optimality conditions enable computing the solution backwards starting at the optimal steady state. One can infer all period t variables from knowing all variables at period $t+1$. For example, from the capital stock in period $t+1$ one infers the interest rate, which using the consumption Euler equation, yields consumption in period t. This in turn allows to infer the level of investment and capital in period t. Credit constraints break this link. Knowing the interest rate and period $t+1$ consumption of households who are credit constrained in period t is not sufficient to infer their period t consumption level; a binding credit constraint prevents us from using the consumption Euler equation as is possible in the complete markets case. As a result there is no deterministic link between the optimal and the initial steady state. Note that, from a computational perspective, this missing link prevents us also from using a simple "backward shooting" algorithm. But, as we explain in Section 5, it is Theorem 2 that renders the computational algorithm tractable, since we can first compute the steady state independent from the transition path and in a second step, solve for the transition path knowing both the initial and terminal conditions.[11]

The intuition for why there is a unique optimal level of government debt is straightforward. As in Barro (1979) and as in complete markets models the planner aims to smooth distortions over time using government debt. But with incomplete markets there is an additional benefit of providing more bonds: better consumption smoothing. The planner therefore deviates from full distortion smoothing and instead faces a trade-off between consumption and distortion smoothing. As a result the planner lowers labor taxes initially as there are two benefits - lower distortions today, and higher debt and thus better consumption smoothing - but just one cost (higher distortions tomorrow). Of course there are limits to how high debt can become as eventually future distortions become too big and outweigh the initial lower distortions and the benefits of higher liquidity. The optimal level of government debt is determined as equalizing the benefits and costs at the margin. As a result, in the long run both labor taxes and government bonds are high, which has the additional advantage

[11] The credit constraints also explain why the optimal steady state wealth distribution is independent from initial conditions. One property of the Aiyagari model is that the credit constraint will be eventually binding for everyone. At the point in time when the credit constraint is binding a household's life is reset and the individual history until this point is wiped out. Eventually everyone's history was eliminated at some point such that the current situation is independent from the initial one, implying that each individual's initial income level will be irrelevant for the long-run income position.

that risky labor income is replaced with safe asset income.

A more formal intuition, and one that is moving us closer to how the proof works, is to note that there are not enough independent optimality conditions to determine the long-run steady state if markets are complete. Government bonds have no net worth since Ricardian equivalence holds in complete markets models, and therefore agents are willing to hold any amount of bonds in steady state. As a result bonds, B, appear only in the government budget constraint (the household budget constraint is dropped by Walras' Law) but this is not sufficient to pin down the long-run level of bonds. The steady-state government budget constraint determines only pairs of B and labor taxes τ_n which satisfy this constraint but does not determine each separately. In other words, an equation is missing. Thus, the long-run level of government debt (and also of labor taxes) is not determined from the steady state FOCs, but only when initial conditions are taken into account.

We now argue that incomplete market models provide an additional equation - the asset demand equation - which serves to determine the long-run debt level since bonds have net worth in this class of models.[12]

As is well known, aggregate households' steady-state asset demand in the Aiyagari economy is described through a mapping between the after-tax interest rate $\bar{r}$ and assets A as illustrated in Figure 1a.[13] Since the capital stock is at its modified golden rule level K^* where the marginal product of capital equals $1/\beta$ (Theorem 1), total assets $A = K^* + B$ are one-to-one related to the number of bonds. Figure 2a shows that picking a specific capital income tax rate and therefore an after-tax interest rate $\bar{r}$, automatically also chooses a spe-

[12]Some intuition can also be gained from a simple reduced-form model where bonds by assumption have value, where the representative agent's utility equals

$$\sum_{t=0}^{\infty} \beta^t (u(c_t) + \chi(B_{t+1}))$$

and the household budget constraint is (inelastic labor $n = 1$)

$$B_{t+1} = (1 + \bar{r}_{t+1})B_t - c_t + w_t.$$

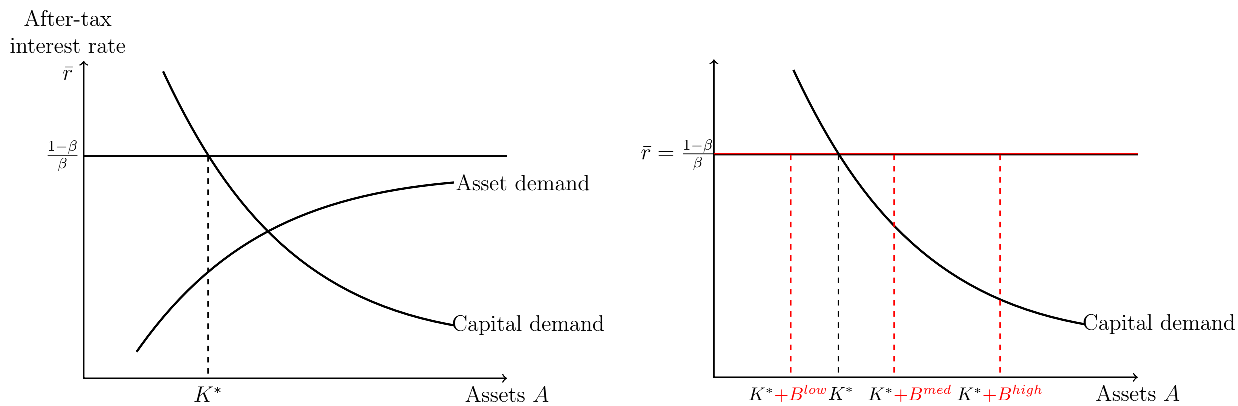
In steady state the planner has to respect households' demand for bonds function,

$$1 - \frac{\chi'(B)}{u'(c)} = \beta(1 + \bar{r}),$$

which is the additional equation that determines the long-run level of bonds in the Ramsey planner problem. The intuition in our incomplete markets model is the same with the important difference that bonds have a real value not by assumption but endogenously.

[13]For a textbook treatment of incomplete market models see Ljungqvist and Sargent (2012).

Figure 1: Asset Markets in (In)complete Markets



(a) Asset Market: Aiyagari Model

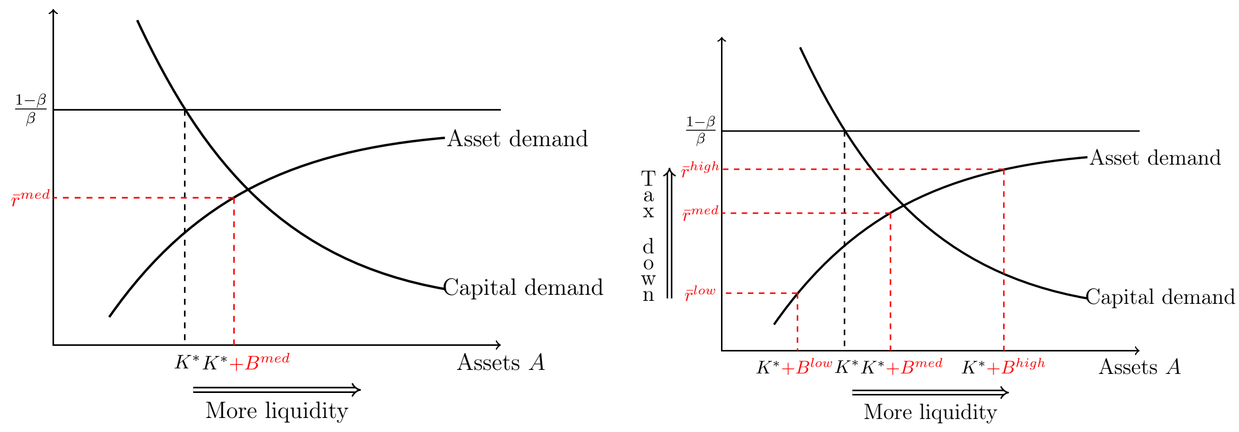
(b) Asset Market: Complete Markets

cific amount of bonds B and vice versa. The planner therefore faces a trade-off, illustrated in Figure 2b, between supplying more bonds/liquidity and lower capital income tax rates. Choosing a low level of bonds, B^{low}, allows for a low after-tax interest rate $\bar{r}^{low}$, that is a high tax on capital income. Choosing higher levels of bonds, B^{med} or B^{high}, provides more liquidity and thus enhances consumption smoothing, but the capital income tax rates have to fall as households require higher after-tax interest rates, $\bar{r}^{med}$ or $\bar{r}^{high}$, to be willing to absorb the higher amount of assets $K^* + B$. This $B - \bar{r}$ trade-off provides the additional equation allowing us to determine the long-run level of debt using just the steady-state FOCs. This trade-off is absent in complete markets models and therefore the long-run level of bonds is not determined as illustrated in Figure 1b. In a steady state $1 + \bar{r} = 1/\beta$ and Ricardian equivalence implies that the representative agent is willing to hold any amount of bonds, $A^{low}, A^{med}, A^{high}$.

The formal proof uses ideas and concepts developed by Debreu (1970) to show the generic local uniqueness of competitive equilibria. We use the same approach since both in Debreu (1970) and here, one has to show that a set of equations is locally invertible and thus has a unique local solution. In Debreu (1970) this set of equations is given by the excess demand function and here it is the set of equations characterizing the optimal steady state. Local uniqueness is guaranteed generically, meaning it holds for a set of parameters of measure one (here, the distribution of idiosyncratic productivity; in Debreu (1970), initial endowments).[14]

[14]The same proof to show local uniqueness can be used to show that the constraint qualification is generically satisfied such that the Karush-Kuhn-Tucker optimality conditions are necessary.

Figure 2: Additional Equation: B-$\bar{r}$ trade-off



(a) Additional Equation

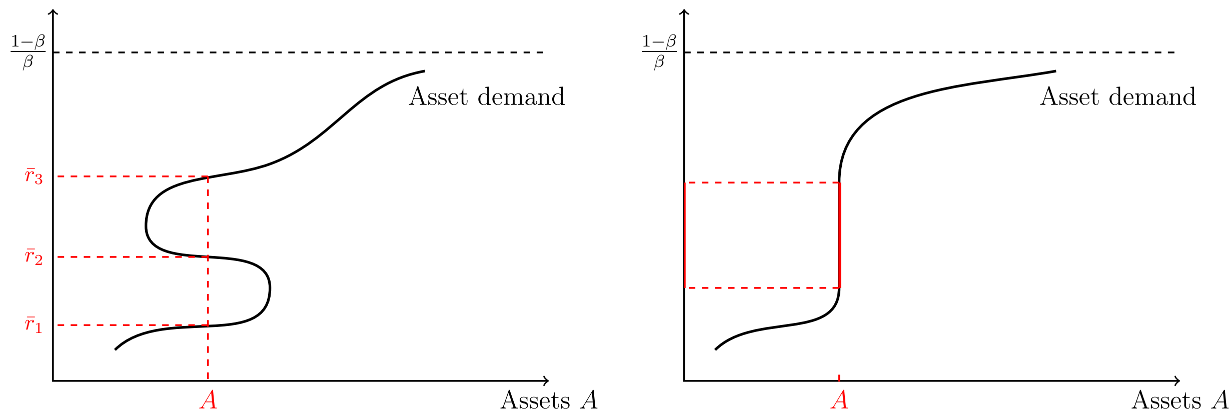
(b) B-$\bar{r}$ Trade-off

As in Debreu (1970) local uniqueness implies that there are at most a finite number of solutions to the necessary FOCs of the optimal steady state. Figures 4 and 3 illustrate this reasoning. The two panels in Figure 4 show the simple case where the asset demand curve is monotonically increasing and therefore each level of assets is associated with a different after-tax capital income tax rate $\bar{r}$. That is, we obtain only one solution. Figure 3a illustrates that a finite number of solutions is possible, that is multiple levels of interest rates, $\bar{r}_1, \bar{r}_2, \bar{r}_3$, are associated with the same asset level A. What both figures have in common is that all solutions are locally unique; can be separated by open sets.[15] Adopting the arguments in Debreu (1970) shows that this is the generic case. Figure 3b shows a non-generic case where a continuum of interest rates $\bar{r}$ is associated with the same A and thus an infinite number of solutions would be possible. Following the arguments in Debreu (1970) we show that this is a pathological case and not robust to small perturbations of fundamentals (distribution of productivity shocks).

Since the steady state depends continuously on initial conditions - such as the initial debt level - the finiteness of the number of steady states implies that the steady state does not depend on initial conditions.

[15]For each solution e there is an open set U_e such that $e \in U_e$ and no other solution is in U_e.

Figure 3: Generic Local Uniqueness



(a) Finite Number of SS (b) Non-generic Indeterminacy

4 Quantitative Analysis: Steady State

The quantitative analysis has two main parts. First, we compute the optimal policy in the long-run in this Section. Second, we use the optimal long-run policy as a terminal condition to compute the optimal policy along the transition path in Section 5.

We start by calibrating the model to the U.S. economy, and then compute the optimal values for the capital and labor tax rates, the capital stock and the level of debt in the steady state.[16] We also compute the optimal policy for a different Frisch elasticity, for a different elasticity of intertemporal substitution, for the income process as used in Aiyagari (1995) (with much lower income inequality than in our calibrated benchmark model) and for a specification of the income process allowing for permanent productivity differences. We also use the same calibrations and solve for the optimal policies when lump-sum transfers are an additional available instrument, obviously a very effective tool for redistribution.

4.1 Calibration

To calibrate the initial steady state of the benchmark economy to the U.S. economy, we first set the initial values of the following variables according to common practice in the literature. Following Trabandt and Uhlig (2011), initial labor income tax rate is set to 28% and capital income tax rate to 36%, as shown in table 1. The debt-to-GDP ratio is

[16]These are the same policy instruments as used in Aiyagari (1995).

Table 1: Benchmark Calibration

Parameters	Value	Description	Source/Target
		Exogenous Parameters	
σ	2	Coefficient of Risk Aversion	
ϕ	1	Frisch Elasticity	
α	0.36	Capital Share	
δ	0.08	Depreciation Rate	
τ_l	28%	Labor Income Tax Rate	Trabandt and Uhlig (2011)
τ_k	36%	Capital Income Tax Rate	
B/Y	62%	Debt to GDP Ratio	Holter et al. (2015)
G/Y	7.3%	Gov. Expenditure to GDP Ratio	Prescott (2004)
		Calibrated Parameters	
ρ	0.93	Persistence of Labor Productivity	$a_{90}/a_{50} = 7.55$
σ_u	0.30	Std. Dev. of Labor Productivity Shock	$var\left(\log y\right) = 1.3$
β	0.94	Discount Rate	$K/Y = 3$
χ	13.4	Disutility from Labor	$mean(n) = 0.33$

62% as in Holter et al. (2015), and government expenditure is 7.3% of GDP, as in Prescott (2004). Then, we set some parameters in the utility function and production function as follows: $\sigma = 2, \phi = 1, \alpha = 0.36$ and $\delta = 0.08$. The values for σ, α and δ are those most commonly used in the literature. The value of the Frisch elasticity ϕ is set higher than what is considered a typical choice in the empirical labor literature, but lower than the choice among many macroeconomists. As this parameter is important for the size of labor taxes in standard models, we provide several robustness checks. Anticipating our result of high labor income taxation in the long run, the relatively high choice of $\phi = 1$ shows that this finding is not due to an inelastic household labor supply.

The remaining of parameters are set to match related targets in the U.S. economy. Idiosyncratic labor productivity evolves according to the AR(1) process $\log e_t = \rho \log e_{t-1} + u_t$, $u_t \sim N\left(0, \sigma_u\right)$, where $\rho = 0.933$ and $\sigma_u = 0.30$ are calibrated to two targets in the U.S. economy: first, the variance of log labor income - 1.3, and second, the ratio of asset holdings at the 90 percentile over asset holdings at the 50 percentile - 7.55. In the benchmark this persistent stochastic process is the only source of individual heterogeneity while we add permanent differences in productivity in the robustness analysis below. The time preference is set as $\beta = 0.938$ to match a capital-output ratio of 3. The disutility from labor is $\chi = 13.4$ such that the labor supply on average is 0.33.

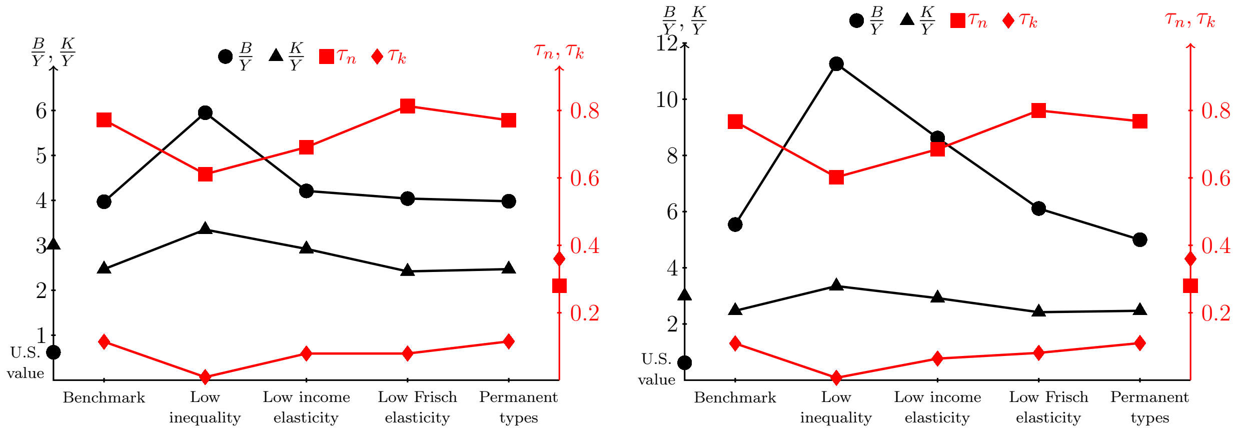
4.2 Results

We solve numerically the set of equations characterizing the steady state of the optimal policy problem - equation (19) to (23). Appendix III.1 describes the computational algorithm. We conduct this experiment for the benchmark calibration shown in section 4.1, and several other parametrizations where we change one parameter at a time. We consider a low Frisch labor supply elasticity of $\phi = 0.5$ instead of 1, a small income effect of $\sigma = 1$ instead of 2 and a low inequality calibration as in Aiyagari (1995) where we set $\rho = 0.6$ and $\sigma_u = 0.2$ instead of 0.93 and 0.3 (we also have to change β and χ to match the benchmark capital output ratio and labor supply). Finally we allow for permanent productivity differences in addition to an stochastic element implying that not all income states can be reached from any other state, e.g. the most productive worker today can never fall to the lowest productivity. While we discuss the results in detail in the following sections, the upper panel of Figure 4a provides an overview of optimal policies given fixed transfers.

While the precise numbers and magnitudes of optimal policy vary across these parameterizations, we obtain several robust substantive conclusions. In the long run, the levels of government debt and labor income taxes are very high and the capital tax is low relative to the current U.S. level. Tax distortions apparently do not put a tight bound on the welfare-maximizing debt level. While a complete explanation requires computing the full transition path to the steady state as we show below, several intuitive arguments can be made only considering the steady state. One line of reasoning is that higher labor taxes reduce the risky income stream and replace it with risk-free capital income from holding bonds. The high level of debt along with the modified golden rule for capital imply that households require a higher after-tax interest rate and thus the tax on capital income is low across parameterizations. The high level of debt also implies large interest rate payments requiring a quite high tax on labor income, again robustly across all calibrations. The results therefore show that the planner does not use high capital income taxes for redistribution, but instead decides to tax the risky labor income at a high rate, and provides safe interest rate income from holding a large amount of debt which serves to smooth consumption very well.

The transition analysis complements these findings and shows that the high steady-state debt level is a consequence of low or even negative labor taxes during the transition. Initially, it is optimal to have low labor distortions and to redistribute toward households which primarily rely on their labor income. This policy is financed through issuing debt, and labor taxes have to be raised eventually to pay for the interest payments on this debt.

Figure 4: Results Overview



(a) Fixed Transfers (U.S. Level) (b) Optimal Transfers

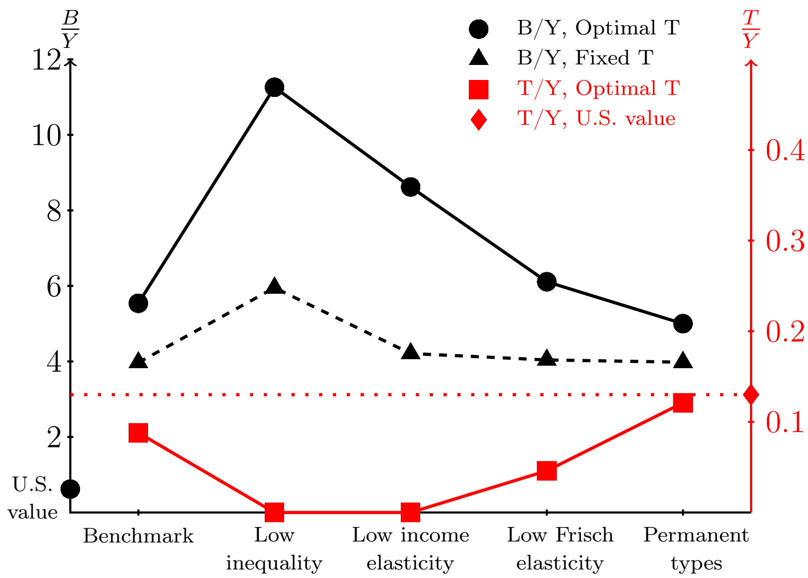
The transition analysis also explains why moving to a high debt, high labor taxes steady state is welfare-enhancing although the welfare in the final steady state is lower than for the calibrated initial steady state. The welfare gains during the transition when debt is accumulated and labor taxes are low outweighs the later welfare losses when labor taxes have to be raised.

Figure 4b shows the same overview of results but now, we allow transfers to be chosen by the planner instead of fixing them at the current U.S. level. We see that the broader conclusions on the optimal level of labor and of capital income taxes rates, of debt and of the capital stock, are largely unchanged when we move from fixed transfers to optimal transfers.

What does vary the most across parameterizations is the the level of debt, which is even higher than the benchmark in some cases, and the optimal level of transfers as Figure 5 shows. Figure 5 also shows that if a different parametrization leads to a higher level of transfers, the level of debt is lowered and vice versa, if the level of transfers is lowered the level of debt is higher.

The detailed descriptions and explanations of results for the benchmark calibration are considered in section 4.2.1, for a low inequality economy in section 4.2.2, for a low Frisch elasticity in section 4.2.3, for a low income elasticity in section 4.2.4 and for permanent productivity differences in section 4.2.5. We do not limit our analysis to the case where transfers are exogenous, but also report results when we include transfers as an instrument

Figure 5: Optimal Transfers and Level of Government Debt



(the details of the numerical approach are delegated to Appendix III).

4.2.1 Results: Benchmark Calibration

The findings for the optimal Ramsey policies for using three instruments (labor tax τ_n, capital tax τ_k and transfers T) in a steady state are summarized in column (1) of Table 2 while the corresponding numbers - calibrated to the U.S. economy - are in column (4), as a comparison. In the long run, the optimal labor tax rate is as high as 76.7%, while the capital tax rate is 10.9% - higher than the optimal tax rate of 0 in a complete markets model, but lower than the current capital tax rate in the U.S. The quite considerable tax income is spent on redistribution through lump-sum transfer - 8.8% of GDP (smaller than in the calibrated benchmark) - and more importantly, on interest payment of the government debt (the debt level is as high as 5.54 times GDP). The capital satisfies the modified golden rule, so the capital output ratio is 2.47, slightly lower than the current ratio in the U.S. The high labor tax reduces labor supply from 0.33 to 0.21. This policy leads to a bigger inequality of labor income but reduces the inequality of wealth.

This steady state features a high tax rate on labor and a large amount of redistribution. First, the social planner shrinks income inequality by setting a high labor tax rate, even

Table 2: Steady-state Ramsey Solutions: Benchmark Economy

Instruments:	(τ_k, τ_n, T) (1)	(τ_k, τ_n) (2)	$(\tau_k, \tau_n, T=0)$ (3)	U.S. Calibration (4)
τ_l	76.7%	77.2%	75.6%	28%
τ_k	10.9%	11.4%	11.2%	36%
T/Y	8.8%	18.3%	0	13.4%
B/Y	5.54	3.97	6.98	0.62
K/Y	2.47	2.47	2.47	3.00
$\int n$	0.21	0.20	0.22	0.33
$N = \int en$	0.27	0.27	0.27	0.33
coeff. var. a	0.70	0.83	0.61	1.51
coeff. var. y	1.70	1.72	1.68	1.26
$var\,(\log y)$	3.05	2.99	3.06	1.30
$var\,(\log(y+T))$	0.45	0.23	3.06	0.48
$var\,(\log(y+T+\bar{r}a))$	0.35	0.30	0.41	0.46

Note - The table contains the optimal Ramsey steady-state policies. (1): Labor tax τ_n, capital tax τ_k and transfers T are available instruments. (2): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (3): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are set to zero. (4): U.S. economy (calibration target)

though given the high Frisch elasticity the distortion on the labor supply is sizable, at 36% lower compared to the level in the calibrated U.S. economy. Effective labor N drops by less (18%), as it is low productivity households who reduce their labor supply most such that the inequality of after-tax labor income $log(y) = \log(\bar{w}en)$ is higher in the optimal solution. However, the planner spends a fraction of the tax income as lump-sum transfer, which reduces inequality of after tax and transfer income $\log(y+T)$ from 0.48 to 0.45, leading to an improvement of low-income households' welfare. The planner lowers inequality further through paying large interest on government debt and reducing wealth inequality. The coefficient of variation drops from 1.51 to 0.70, which manifests in lower inequality of income $\log(y + T + \bar{r}a)$ of 0.35 relative to the benchmark level of 0.46.

To better understand the importance of lump-sum transfers in redistribution we now consider the same optimal policy problem with one modification: either we fix the size of transfers at their current level or do not allow the planner to use lump-sum transfers and set $T = 0$. The findings are reported in columns (2) and (3) of Table 2. When $T = 0$ is enforced (column (3)), the optimal labor tax rate is slightly lower, 75.6% and the capital tax rate is 11.2%. Now that the government pays no transfer, the still high revenue from taxing labor income is spent on the interest on huge government debt, which increases to 6.98 times GDP, even larger than the debt level of 5.54 for the case with transfers. Households on average hold high levels of assets and their returns are an important source of income. The asset returns help insure against the negative labor productivity shocks, especially in this case where asset inequality is low - the coefficient of variation of household assets is only 0.61, compared to 1.51 in the original calibration. This means that even some labor income poor households hold sizable assets, and receive substantial interest payments on government bonds. Although they no longer receive direct transfers from the government, the indirect transfers through a more equal distribution of wealth substitute for the absence of the (more effective) redistribution through transfers. As a result the inequality of $\log(y+T+\bar{r}a) = 0.41$ is larger than when transfers payments are allowed for, but smaller than in the benchmark. Yet the variance of $\log(y) = \log(y + T) = 3.06$ is much higher than in the benchmark, since labor supply again is more unequal. If we fix transfers at the current U.S. level of 13.4% of current U.S. GDP, higher than the optimal value reported in column (1), labor supply decreases even further but now the optimal level of debt drops to about 4 times GDP (column (2) of Table 2). Now that transfers already ensure considerable redistribution, less redistribution and insurance through government debt are needed.

4.2.2 Results: A Low Inequality Economy

The income process in our benchmark calibration implies a large amount of inequality. If instead we target an economy with lower income inequality, as in Aiyagari (1995), the motive for redistribution is smaller and some of the optimal policy choices are quite different. Based on the benchmark economy, we change the parameters of the income process to $\rho = 0.6$ and $\sigma_u = 0.2$. Then we recalibrate the model by changing β to 0.973 and χ to 12.8 to match a capital output ratio of 3 and an average labor supply of 0.33.

The optimal long-run tax rates in this low inequality economy are lower, compared to the benchmark economy. As shown in Table 3, the labor income tax rate drops to 60.2% though it is still quite high, compared to the level in the current U.S. economy. The capital income tax rate is close to 0, only 0.7%. The transfer is 0 (rounded) and the level of government debt is even larger than in the benchmark, at 11 times GDP.

In this low inequality economy, the planner has, compared to the high inequality benchmark economy, fewer incentives to reduce the income risks and redistribute to low-income

Table 3: Steady-state Ramsey Solutions: Low Inequality Economy, $\rho = 0.6, \sigma_u = 0.2$

Instruments:	(τ_k, τ_n, T) (1)	(τ_k, τ_n) (2)	U.S. Calibration (3)
τ_l	60.2%	61.1%	28%
τ_k	0.7%	0.9%	36%
T/Y	0	15.0%	13.4%
B/Y	11.26	5.95	0.62
K/Y	3.35	3.35	3.00
$\int n$	0.29	0.28	0.33
$N = \int en$	0.30	0.30	0.35
coeff. var. a	0.51	0.57	0.71
coeff. var. y	0.65	0.60	0.51
$var\,(\log y)$	0.38	0.33	0.24
$var\,(\log(y+T))$	0.38	0.11	0.13
$var\,(\log(y+T+\bar{r}a))$	0.09	0.06	0.10

Note - The table contains the optimal Ramsey steady-state policies for the low inequality economy, $\log e_t = \rho \log e_{t-1} + u_t$ with $\rho = 0.6$ and $\sigma_u = 0.2$. (1): Labor tax τ_n, capital tax τ_k and transfers T are available instruments. (2): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (3): U.S. economy (calibration target)

households, resulting in a lower labor tax rate and zero transfers. Meanwhile, the planner still makes large payments to households, through the interest payments on government debt, which helps to insure against the labor productivity shocks, as in the benchmark case with zero transfers.

Our findings show how a welfare-maximizing planner should redistribute and make use of two instruments. One instrument is to pay lump-sum transfers, the classic way to redistribute. In an incomplete markets setting, the government has a second option, as the asset-income of households can be increased. This means of redistribution involves issuing government debt while keeping the steady-state capital fixed at the level that satisfies the modified golden rule.

Which instrument should the planner use, transfers or government debt? It turns out that the answer to this question depends on the labor supply elasticity and the amount of inequality. Using transfers is more effective in redistributing from high- to low-income households but comes with a disadvantage: it also reduces labor supply through income effects. As a result, transfers are used when inequality is high (and thus the need to redistribute is high). Debt is more effective than paying transfers when inequality is low, since labor supply is reduced far less than when transfers are paid. The use of transfers or debt for redistribution also depends on the labor supply elasticity: the choice to use a higher elasticity implies lower transfers and more debt ceteris paribus. We will discuss this in more detail below.

4.2.3 Results: Low Labor Supply Elasticity

Now we consider a lower labor supply elasticity and set the Frisch elasticity to $\phi = 0.5$. Both labor supply and labor income become more volatile due to the drop in the Frisch elasticity, requiring an adjustment to the stochastic process for productivity. As a result of this recalibration to match the same targets as in the benchmark, we now use $\beta = 0.936$, $\rho = 0.926$, $\sigma_u = 0.34$ and $\chi = 37.5$. The modified golden rule capital stock computed using the recalibrated parameter values is close to the one in the benchmark. The steady-state results are reported in Table 4.

A lower elasticity of labor supply implies that labor supply is less sensitive to an increase in labor taxes, rendering a labor tax of 80% optimal. At the same time, although the steady-state labor tax rate is higher than in the benchmark economy, the labor supply is in fact slightly higher: 0.23, compared to 0.21 in the benchmark. This is as expected when the labor supply elasticity is reduced.

4.2.4 Results: Small Income Effect

How labor supply reacts to the labor tax depends not only on the Frisch elasticity which governs the substitution effect, but also on the coefficient of relative risk aversion, which determines the income effect. To explore the role of income effects in shaping our results, we consider a smaller income effect by setting the coefficient of relative risk aversion $\sigma = 1$. As a result of a recalibration we use now $\beta = 0.958$, $\rho = 0.901$, $\sigma_u = 0.29$ and $\chi = 4.6$, so that the modified golden rule capital stock is now larger than in the benchmark. Results are reported in Table 5.

A larger income effect increases labor supply, because a higher labor tax makes households poorer and thereby increases their labor supply, holding other things constant. The smaller income effect in this experiment implies that labor supply is more responsive to labor taxes, rendering a lower labor tax rate of 68.5% optimal. The smaller income effect also decreases the negative income effects on labor supply of paying higher transfers, suggesting that a higher level of transfers than in the benchmark might be optimal. However, this argument

Table 4: Steady-state Ramsey Solutions: Low Labor Supply Elasticity, $\phi = 0.5$

Instruments:	(τ_k, τ_n, T) (1)	(τ_k, τ_n) (2)	U.S. Calibration (3)
τ_l	80.0%	81.3%	28%
τ_k	8.1%	7.9%	36%
T/Y	4.6%	18.1%	13.4%
B/Y	6.11	4.04	0.62
K/Y	2.42	2.42	3.00
$\int n$	0.23	0.22	0.33
$N = \int en$	0.28	0.28	0.33
coeff. var. a	0.63	0.79	1.53
coeff. var. y	1.64	1.68	1.32
$var\,(\log y)$	2.20	2.19	1.30
$var\,(\log(y+T))$	0.59	0.18	0.51
$var\,(\log(y+T+\bar{r}a))$	0.37	0.29	0.51

Note - The table contains the optimal Ramsey steady-state policies for the low labor supply elasticity, $\phi = 0.5$. (1): Labor tax τ_n, capital tax τ_k and transfers T are available instruments. (2): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (3): U.S. economy (calibration target)

overlooks that a smaller income effect also reduces the welfare gains from redistribution. Households are less averse to consumption fluctuations in this case, suggesting a smaller level of transfers than in the benchmark. Our results show that the latter effect dominates the first and transfers are now zero (rounded) in the optimal steady state. But we can conclude that our finding of a high tax on labor income is robust with respect to what we assume for the elasticity of labor supply and the income effect.

4.2.5 Results: Permanent Income Differences

In this experiment, in addition to the labor productivity shocks described in the benchmark, we introduce permanent differences in labor productivities. There are two types of households, with the more productive type 20% more productive than the other. More specifically, there are both time-varying and time invariant components in the labor productivity of a household. In the more productive type's log labor productivities, the time-invariant fixed effect is 0.2, while it is 0 for the other type. The time-varying labor productivity of a household follows an AR(1) process like in the experiments above. In the recalibrated model, the parameters are very close to values in the benchmark: $\beta = 0.939$, $\rho = 0.933$, $\sigma_u = 0.3$,

Table 5: Steady-state Ramsey Solutions: Small Income Effect, $\sigma = 1$

Instruments:	(τ_k, τ_n, T) (1)	(τ_k, τ_n) (2)	U.S. Calibration (3)
τ_l	68.5%	69.1%	28%
τ_k	6.4%	7.9%	36%
T/Y	0	18.5%	13.4%
B/Y	8.62	4.21	0.62
K/Y	2.92	2.92	3.00
$\int n$	0.23	0.22	0.33
$N = \int en$	0.32	0.31	0.42
coeff. var. a	0.69	0.94	1.50
coeff. var. y	1.62	1.63	1.35
$var\,(\log y)$	2.01	1.94	1.30
$var\,(\log(y+T))$	2.01	0.27	0.51
$var\,(\log(y+T+\bar{r}a))$	0.53	0.31	0.49

Note - The table contains the optimal Ramsey steady-state policies for small income effects, $\sigma = 1$. (1): Labor tax τ_n, capital tax τ_k and transfers T are available instruments. (2): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (3): U.S. economy (calibration target)

$\chi = 13.4$.

The optimal policies also are very similar to the benchmark values. If the transfer is fixed at the initial level, the optimal labor and capital taxes are 77.1% and 11.5% - almost identical to the solution in the benchmark. Other model moments - government debt, labor supply, income and asset inequalities - also are almost the same as in the optimal steady state in the benchmark calibration. If the transfer is optimally chosen by the government, it is a bit higher than in the benchmark, whereas labor and capital taxes are slightly higher and government debt is lower. This difference in the optimal policies is explained by the permanent income differences: given that one type of households is permanently less productive than the other, the government prefers to redistribute from the more productive type to the less productive one, using a higher level of transfers. The transfer is then funded by lower government debt and slightly higher taxes. With the higher level of transfers, the inequality of wealth rises a bit compared to the benchmark, the inequality of income including transfers is lower, and the inequality of total income, from labor, transfers and asset returns, is similar to the benchmark level.

Table 6: Steady-state Ramsey Solutions: Permanent Income Differences

Instruments:	(τ_k, τ_n, T) (1)	(τ_k, τ_n) (2)	U.S. Calibration (3)
τ_l	76.8%	77.1%	28%
τ_k	11.0%	11.5%	36%
T/Y	12.1%	18.3%	13.4%
B/Y	5.00	3.98	0.62
K/Y	2.47	2.47	3.00
$\int n$	0.21	0.20	0.33
$N = \int en$	0.27	0.27	0.33
coeff. var. a	0.74	0.83	1.51
coeff. var. y	1.71	1.72	1.26
$var\,(\log y)$	3.02	2.97	1.30
$var\,(\log(y+T))$	0.34	0.23	0.48
$var\,(\log(y+T+\bar{r}a))$	0.34	0.30	0.47

Note - The table contains the optimal Ramsey steady-state policies for permanent differences in productivity. (1): Labor tax τ_n, capital tax τ_k and transfers T are available instruments. (2): Labor tax τ_n and capital tax τ_k are available instruments. Transfers T are fixed at the benchmark level. (3): U.S. economy (calibration target)

4.2.6 Understanding High Labor Taxes

The optimal policies feature very high labor tax rates and large redistributions with adverse consequences for aggregate employment, a reduction by 36%. Effective labor supply is, however, reduced by less (18% in the benchmark economy) since the reduction in hours worked is concentrated on low productivity households.

We now conduct a simple decomposition to understand what drives this drop in aggregate steady-state effective labor supply from the initial level

$$N = \int_{E\times A} en(a,e,s(a,e;\tau_n,T);\tau_n,T)d\mu(a,e), \tag{26}$$

to the lower level in the optimal policy steady state,

$$N^* = \int_{E\times A} en^*(a,e,s^*s(a,e;\tau_n^*,T^*);\tau_n^*,T^*)d\mu^*(a,e). \tag{27}$$

Individual labor supply of a household with asset level a and labor productivity e, who saves $s(a,e;\tau_n,T)$ and faces the labor tax rate τ_n and the transfer T in the initial steady state is

$n(a, e, s(a, e; \tau_n, T); \tau_n, T)$. In the optimal policy steady state labor supply of a household with asset level a and labor productivity e changes to $n^*(a, e, s^*(\cdot); \tau_n^*, T^*)$ for a different labor tax rate τ_n^* and different transfers T^*. In the new steady state households' saving behavior also changes to a new function $s^*(\cdot)$ which in turn affects labor supply. The reason why the labor supply functions n and n^* are different conditional on taxes, transfers and savings is that prices, the before-tax wage and the after-tax interest rate, change too and affect labor supply. The effect of these prices changes is captured through a change from n to n^*. The labor supply functions are then integrated using the different stationary distributions μ and μ^* to yield the aggregate labor supplies N and N^*.

The change in aggregate supply between the optimal steady state, N^*, and the initial calibrated steady state, N, can then be decomposed into the contribution of the changes in labor taxes, in transfers, in prices, in savings and in the joint distribution of assets and shocks

$$
\begin{aligned}
N^* - N &= \underbrace{\int_{E\times A} en^*(a, e, s^*(\cdot); \tau_n^*, T^*) d\mu^*(a, e) - \int_{E\times A} en(a, e, s(\cdot); \tau_n, T) d\mu(a, e0)}_{\text{Total effect} = -0.0546\ [-0.1204]} \qquad (28) \\
&= \underbrace{\int_{E\times A} en(a, e, s(a, e; \tau_n, T); \boldsymbol{\tau_n^*}, T) d\mu(a, e) - \int_{E\times A} en(a, e, s(a, e; \tau_n, T); \boldsymbol{\tau_n}, T) d\mu(a, e)}_{\text{Labor tax change} = 0.1147\ [0.0028]} \\
&+ \underbrace{\int_{E\times A} en(a, e, s(a, e; \tau_n, T); \tau_n^*, \boldsymbol{T^*}) d\mu(a, e) - \int_{E\times A} en(a, e, s(a, e; \tau_n, T); \tau_n^*, \boldsymbol{T}) d\mu(a, e)}_{\text{Change in Transfer} = 0.0573\ [0.1078]} \\
&+ \underbrace{\int_{E\times A} e\boldsymbol{n^*}(a, e, s(a, e; \tau_n, T); \tau_n^*, T^*) d\mu(a, e) - \int_{E\times A} e\boldsymbol{n}(a, e, s(a, e; \tau_n, T); \tau_n^*, T^*) d\mu(a, e)}_{\text{Change in Prices} = -0.0641\ [-0.0407]} \\
&+ \underbrace{\int_{E\times A} en^*(a, e, \boldsymbol{s^*(a, e; \tau_n^*, T^*)}; \tau_n^*, T^*) d\mu(a, e) - \int_{E\times A} en^*(a, e, \boldsymbol{s(a, e; \tau_n, T)}; \tau_n^*, T^*) d\mu(a, e)}_{\text{Change in Savings Function} = -0.0702\ [0.0226]} \\
&+ \underbrace{\int_{E\times A} en^*(a, e, s^*(a, e; \tau_n^*, T^*) d\boldsymbol{\mu^*}(a, e) - \int_{E\times A} en^*(a, e, s^*(a, e; \tau_n^*, T^*); \tau_n^*, T^*) d\boldsymbol{\mu}(a, e),}_{\text{Change in Distribution} = -0.0923\ [-0.2129]}
\end{aligned}
$$

where for each step we report the change in effective labor supply as well as the change in raw hours in parentheses. Optimal policy variables have a superscript $*$. This decomposition shows that the changes in raw and effective labor not only differ in magnitude but can even have the opposite sign. The differences are explained by the heterogeneity in the labor supply responses of households with different labor productivity and asset levels. For each step in the decomposition, the aggregate changes both in raw and effective labor are driven by low-assets households whose labor supply is high, while for high asset households labor supply is quite low and thus irrelevant for the aggregate changes.

The labor tax change keeps the labor supply function n and the savings function s, as well as prices and the distribution, fixed and is thus a combination of substitution and income effects. The surge in the labor tax rate from 28% to 76.7% increases raw labor supply only a little, by 0.0028, since the income effect and the substitution effect cancel each other out on average. The small aggregate raw labor effect, however, masks some significant heterogeneity of low-asset households' labor supply responses across different productivity levels. The effective labor supply increases by 0.1147 in this experiment. This is because the labor supply of high-productivity households with low assets increases due to income effects outweighing substitution effects, whereas labor supply of low-assets/low-productivity households falls due to substitution effects being stronger than income effects for this group. This heterogeneity in the relative strength of income and substitution effects leads to a small increase in raw labor and a large increase in effective labor when aggregated.

Transfers affect (effective) labor supply through an income effect due to a change in consumption. To isolate the effects of a change in transfers, therefore we, keep the saving function constant so that consumption fully absorbs the change in disposable income. The resulting increase in effective labor supply by 0.0573 thus is due to lower transfers and the induced decrease in consumption which by the income effect increases labor supply. Raw labor supply increases by more, 0.1078, since income effects are the strongest for low-assets/low-productivity households which increase their labor supply the most.

The effect of before-tax wages and after-tax interest rates on labor supply results in different labor supply functions n and n^*, reducing effective labor supply by -0.0641. The lower before-tax wage increases effective labor supply by the same arguments given above for the increase in labor taxes. Just by a smaller magnitude, 0.0161, since the wage change is much smaller than the tax change. The increase in the after-tax interest rate leads to higher asset income, and thus to higher consumption, and through the income effect to lower labor supply, -0.802.

Households also adjust their savings behavior in response to the policy change, leading to a decrease in effective labor supply by -0.0702 while raw labor increases by 0.0226. In the optimal steady state, high-productivity/low-asset households save less and thus work less. Low-productivity/low-assets households save slightly more due to higher after-tax real interest rates and thus work more. The implication here is that an income effect explains the increase in aggregate raw labor supply and the decrease in aggregate effective labor supply.

Finally, the change in the distribution from μ to μ^* has the largest effect on aggregate raw labor supply, reducing it by -0.2129. While the distribution of productivity is unchanged, households on average hold more assets in the optimal policy steady state as the private sector has to absorb an increase in B/Y from 0.62 to 5.54. Labor supply is falling in the level of assets a household holds, so that this large shift in the distribution of assets causes a large shift in aggregate labor supply. It is important to note that this is a comparison of two distributions with very different mean assets holdings, and it does not describe the labor supply effects of increasing assets a for a fixed distribution.[17] All households provide less hours but high-productivity households to a smaller extent since the income reduction of a cut in hours is proportional to productivity and thus more costly for high-productivity households.

This decomposition shows that the raw labor supply effects are explained mainly by income effects due to higher asset holdings and higher asset income. This shift in asset income has negative income effects on hours supply which are not fully undone by the reduction in transfers and the associated positive income effects. The income effects of the labor tax changes, on the other hand, are largely nullified by substitution effects. The aggregate change in raw labor carries over to effective labor but in a muted way, due to productivity heterogeneity.

5 Quantitative Analysis: Transition

Let us reiterate a main objective, and huge computational challenge, in this paper: to compute the path of tax rates and government debt that maximizes welfare at date 0. To achieve this, several hundred or thousands of variables must be chosen in a highly nonlinear

[17] The latter fixed distribution experiment would combine the disincentive effects of higher assets a with the positive effects of higher productivity. Since productivity and assets are positively correlated, the resulting drop in labor supply would be smaller than when comparing two distributions that differ in their mean asset holdings.

optimization problem. However, our previous result on the optimal steady state turn this non-manageable optimization problem into a manageable one. We have shown that the optimal steady state is independent of initial conditions. From a computational standpoint this means that we know the optimal long-run policies and allocations without having to compute the transition. We know the initial conditions (economy calibrated to U.S. economy) and we know the terminal condition, the optimal steady state characterized above. The computational problem is then to find the optimal policy path that satisfies all necessary first-order conditions along the transition, and at the same time satisfies the initial and terminal conditions. This problem is still very daunting as it involves solving hundreds or thousands of nonlinear equations, but it is significantly easier (and therefore tractable) than the original problem of trying to find the optimal transition and the optimal terminal point at the same time. Given the large number of variables involved in this task there is no way to check the global validity of a candidate solution. However, with our approach this check is not necessary.

Knowing the optimal path of policies allows us to compute the welfare gains of switching to the optimal policy and helps us understand the properties of the optimal steady-state policies better, since those obviously depend on the transition.

5.1 Computational Algorithm

Appendix III.2 outlines the details of computing an optimal transition, starting from the model calibrated to the U.S. economy and going to the optimal long-run steady state.

5.2 Calibration of Initial Steady State

We choose the simulation period to be 800 years since the distribution of assets and productivity converges only slowly. In order to facilitate the computation of the optimal transition path we recalibrate the model with 10-year periods. We have to recalibrate the model, and choose targets in our 10-year period economy to be consistent with those in the 1-year economy in the benchmark economy in Section 4. To choose the parameters governing the productivity process, we run a Monte Carlo simulation to obtain the variance of the log of 10-yearly earnings in the 1-year calibration exercise we performed in Section 4.1. We then adjust σ_u, the standard deviation of the innovations to labor productivity, and the persistence of labor productivity, ρ, to match the variance of this log of 10-yearly earnings,

Table 7: Benchmark Calibration with 10-year Periods

Parameters	Value	Description	Source/Target
		Exogenous Parameters	
σ	2	Coefficient of Risk Aversion	
ϕ	1	Frisch Elasticity	
α	0.36	Capital Share	
δ_{10}	0.72	10-year Depreciation Rate	
τ_l	28%	Labor Income Tax Rate	Trabandt and Uhlig (2011)
τ_k	36%	Capital Income Tax Rate	
B/Y	62%	Debt to GDP Ratio	Holter et al. (2015)
G/Y	7.3%	Gov. Expenditure to GDP Ratio	Prescott (2004)
		Calibrated Parameters	
ρ	0.61	Persistence of Labor Productivity	$a_{90}/a_{50} = 7.55$
σ_u	0.66	Std. Dev. of Labor Productivity Shock	$var\,(\log y_{10}) = 0.95$
β	0.59	Discount Rate	$K/Y = 3$
χ	50.2	Disutility from Labor	$mean(n) = 0.33$

$var(log(y_{10})) = 0.947$, and the ratio between the asset holdings of a household at the 90th percentile and a household at the 50th percentile, $a_{90}/a_{50} = 7.55$. With a 10-year period we set $K/Y = 0.3$ consistent with a capital output ratio of three when Y is annual and not the 10 times higher, 10-year output. Then to get the same annual interest rate of 4% as in section 4.1, $r_{10} = 1.04^{10} - 1$, we adjust the 10-year depreciation rate to $\delta_{10} = 0.720$.

5.3 Results: Transition

We now compute the transition path with optimal choices of debt levels, of capital tax and labor income tax rates. Transfers are kept at the calibrated benchmark level. Figure 6 plots the optimal transition path of capital taxes, τ_k, and labor income taxes, τ_l, and Figure 7 plots the optimal path of the capital stock, K, and government debt, B. It is optimal to subsidize labor in the first few periods and to finance this by increasing debt and taxing capital highly. Low initial labor taxes reduce distortions and increase the welfare of wealth-poor households. Through high initial taxes on capital the planner achieves redistribution, whereas debt issuance relaxes households' credit constraints. The labor income tax, τ_l, starts out at -23% and gradually increases to a level of about 75% after 160 years (the long-run steady-state level of τ_l is 75% with the 10-year calibration). The low or even negative initial taxation of labor income leads to an accumulation of debt to its new high steady-state level of 0.28 (2.8 in yearly terms). This high debt level requires high labor taxes, implying that the long-run steady-state resulting from the path of optimal policy has lower welfare than the initial one. The transition explains why this policy change nevertheless is welfare-enhancing. The welfare gains realized during the early period of the transition outweigh the welfare losses later on.

As is well known in the literature, the planner would most likely want to impose a very high tax on capital in the first period, because capital in the first period is perfectly inelastic. With transfer as an instrument, the planner would like to confiscate the entire capital stock in period one and redistribute. In our case, with labor tax, capital tax and debt as the instruments this is not necessarily the case. However, we impose an upper bound of $\tau_k = 100\%$. The capital tax thus starts out at 100% in period one and gradually decreases to a level of about 11% after 180 years (the long-run steady-state level of τ_k is 12% with the 10-year calibration).

Debt is rapidly increasing in the first few periods and reaches a peak after 80 years before it starts decreasing towards the steady-state level. At the peak, debt relative to GDP in the initial steady state, B/Y_0, is about 0.32 (3.2 in yearly terms), whereas the long-run level is about 0.28 (2.8 in yearly terms). The capital stock decreases smoothly towards the long-run level for the first 100 years and then stays relatively constant.

After 200 years, policies and other variables become very close, and eventually converge, to their long-run steady-state levels.

5.4 Welfare Analysis

To quantify the welfare gain of the optimal transition relative to the initial steady state, we compute the consumption equivalent gain, i.e., by what percentage we need to increase the consumption of all households in all periods and all states given the initial steady-state policies such that their aggregate expected lifetime utilities at period 0 equal the aggregate expected lifetime utilities of households at period 0 given the optimal transition. More specifically, we denote the consumption equivalent gain as φ, and then the expected lifetime utility of a household with asset a_0 and labor productivity e_0, given the consumption

Figure 6: Optimal Transition Path for labor tax τ_l (left) and capital tax τ_k (right)

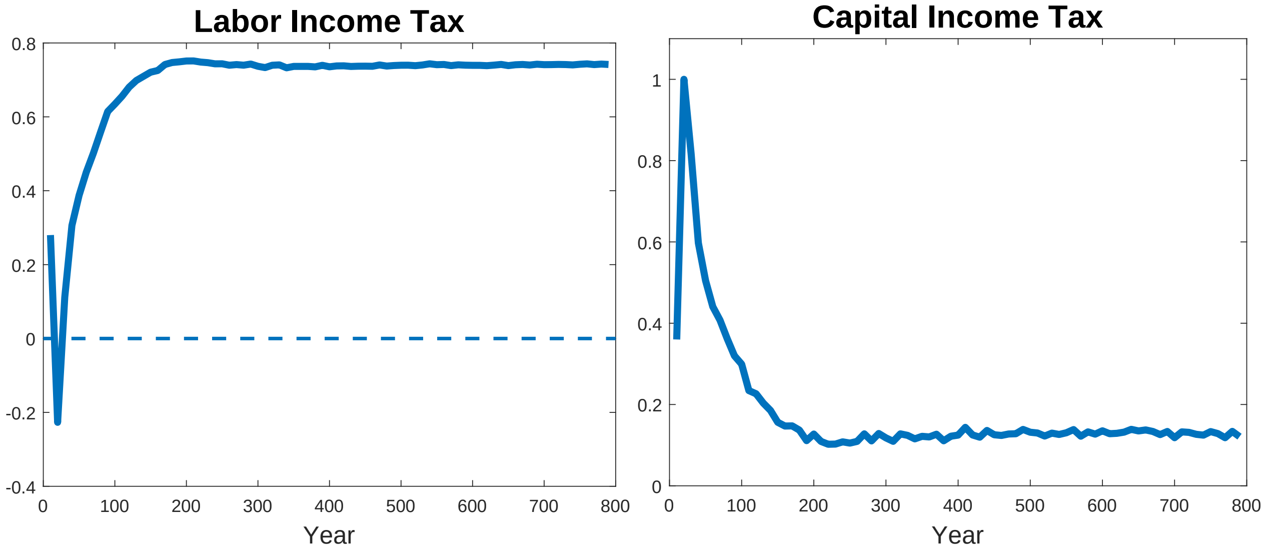
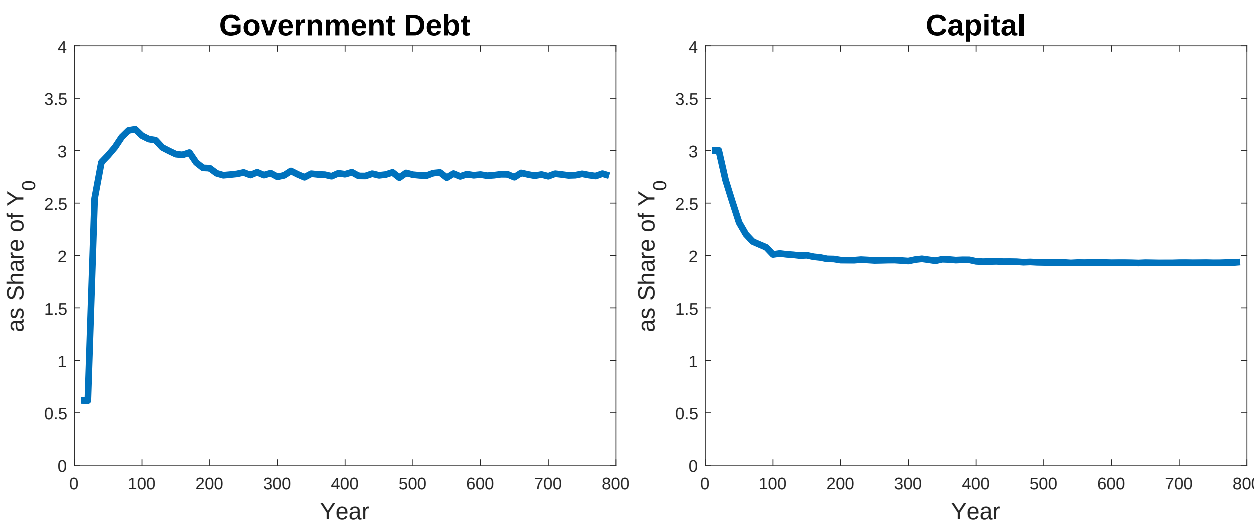


Figure 7: Optimal Transition Path for B (left) and K (right). Y_0 is 10-year output.



equivalence gain and initial steady-state policies, becomes

$$V_0^{ce}(a_0, e_0) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{\left(\varphi c_t^{ini}\right)^{1-\sigma}}{1-\sigma} - \chi \frac{\left(n_t^{ini}\right)^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}} \right),$$

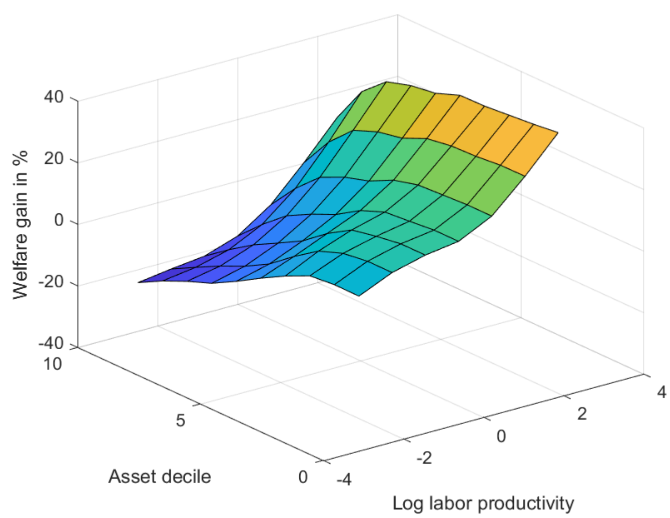
where c_t^{ini} and n_t^{ini} stand for the consumptions and labor supplies of this household from period 0 on and in different periods and states, given the initial policies. Then we solve for the consumption equivalence gain φ, such that the period 0 aggregate welfare of all households, given the consumption equivalence gain and the initial policies, equals the corresponding aggregate welfare given the optimal transition:

$$\int V_0^{ce}(a_0, e_0) d\mu_0 = \int V_0^{opt}(a_0, e_0) d\mu_0, \tag{29}$$

where $V_0^{opt}(a_0, e_0)$ is the expected lifetime utility from period 0 of the household given the optimal transition we computed in this section; recall that μ_0 represents the initial distribution of households.

The lifetime welfare gain achieved from moving to the optimal policy taking into account the full transition period is equivalent to increasing the consumption in all states of the initial steady state by 5.3%. This is a substantial welfare gain, especially because we here have 10-year time periods, and the welfare gain must be viewed as an average welfare gain over 10 years.

Figure 8: Welfare Gain in Consumption Equivalents by Productivity Level and Wealth Decile



To understand who gains and who loses from the optimal tax reform, we can also solve Equation (29) for different parts of the state space. In Figure 8 we plot the welfare gain in consumption equivalents by log labor productivity, $\log e$, and asset decile. As can be seen from the figure the welfare gain is decreasing in the asset level and increasing in labor productivity. This is as we would expect. The initial capital tax of 100% will hurt everyone, but wealthy households the most. The high capital tax thus achieves some redistribution but at the cost of reduced insurance (households have lower asset levels). The initial subsidies to labor are on the other hand welfare-improving for everyone. The gains are largest for more productive households. What the social planner in this economy is lacking is, of course, a progressive labor tax. With only flat labor taxes, the planner cannot help the least productive households without also helping the most productive ones.[18]

6 Conclusion

In incomplete markets models of the Bewley-Imrohoroglu-Huggett-Aiyagari type, inequality is, to a large degree, purely happenstance and this calls for considerable redistribution in an optimal welfare-maximizing policy. Several classic instruments are available: labor income taxation, capital income taxation, and payment of transfers. However, massive redistribution could also inflict big efficiency losses, curbing some of its desirability. While all of these instruments can reduce inequality, what is then unclear is how much redistribution should come from which instrument. Each comes with efficiency losses in terms of distorting labor supply and/or capital accumulation. Furthermore, if markets are incomplete, the planner can also reduce wealth inequality through issuing more debt such that low-labor income households can also rely on their asset income for consumption purchases.

This paper offers the following conclusions on how to redistribute in a welfare maximizing way. The optimal policy to provide insurance is to tax labor heavily in the long run, and redistribute through transfers and government bonds. In particular redistribution works through high labor taxation with capital income taxed only at a low rate, a conclusion that holds in high- and low-inequality economies and is robust to changing parameters such as labor supply elasticity.

The choice of whether to use transfers or higher debt, however, depends on the properties of the economy. Paying transfers is an effective redistributive tool when income inequality is high, but not if inequality is low since then, the disincentive effects on labor supply outweigh the gains from redistribution. In the latter case, a large amount of government debt instead is used to lower wealth inequality and thus make consumption more equal across households. Given this role and the smaller disincentives effects from debt than from paying transfers, the debt level generally is high. This is so not only in low-inequality economies but also

[18]Optimal progressive labor income taxes as in Heathcote et al. (2017) is an interesting extension left for future work.

in high-inequality economies because redistributing through transfers only would inflict too large disincentive effects on the economy.

The results during the transition to the long-run optimum are, however, quite different. During the transition debt is accumulated, and the increase in government revenue is used to bring labor taxes down to below their current U.S. level. Only when the long-run steady state is approached, and the amount of debt and associated interest rate payments are high, does it become necessary to increase labor taxes to balance the budget. At that time, capital taxes will have converged also to a low level after initial periods of high taxation, a well known result as capital is supplied quite inelastically in the short-run.

We prove two theoretical results which enable this quantitative analysis. We show theoretically that the optimal capital stock is at the modified golden rule and that the long-run optimal steady state is independent of initial conditions. In particular, there is a unique long-run optimal level of government debt that is independent of the initial level of debt in our incomplete markets model, a result not valid in complete markets models.

The independence of initial conditions result renders tractable a quantitative analysis of the dynamic optimal taxation problem. This is the first paper to apply a Lagrangian maximization approach to study optimal taxation in an Aiyagari economy. Since we can compute the terminal point first, we are able to design a feasible computational algorithm for finding the entire sequence of optimal policies.

References

Acemoglu, D. and M. K. Jensen (2015): "Robust Comparative Statics in Large Dynamic Economies," *Journal of Political Economy*, 123, 587–640.

Aiyagari, R. S. (1994a): "Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting," Working Paper No. 508, Federal Reserve Bank of Minneapolis.

Aiyagari, S., A. Marcet, T. Sargent, and J. Seppälä (2002): "Optimal Taxation without State-Contingent Debt," *Journal of Political Economy*, 110, 1220–1254.

Aiyagari, S. R. (1994b): "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 109, 659–684.

——— (1995): "Optimal Capital Income Taxation With Incomplete Markets, Borrowing Constraints, and Constant Discounting," *Journal of Political Economy*, 103, 1158–1175.

Aiyagari, S. R. and E. R. McGrattan (1998): "The Optimum Quantity of Debt," *Journal of Monetary Economics*, 42, 447–469.

Aliprantis, C. D. and K. Border (2006): *Infinite Dimensional Analysis. A Hitchhiker's Guide*, Springer-Verlag Berlin Heidelberg, 3 ed.

Barro, R. J. (1979): "On the Determination of the Public Debt," *Journal of Political Economy*, 87, 940–971.

Bhandari, A., D. Evans, M. Golosov, and T. Sargent (2015): "Taxes, Debts, and Redistributions with Aggregate Shocks," Working paper.

——— (2016a): "Fiscal Policy and Debt Management with Incomplete Markets," *forthcoming Quarterly Journal of Economics.*

——— (2016b): "Public Debt in Economies with Heterogeneous Agents," Working paper.

——— (2017): "Optimal Fiscal-Monetary Policy with Redistribution," Working paper.

Carrol, C. D. (2006): "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems," *Economic Letters*, 91, 312–320.

Chamley, C. (1986): "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," *Econometrica*, 54, 607–622.

Chari, V. V. and P. J. Kehoe (1999): "Optimal Fiscal and Monetary Policy," in *Handbook of Macroeconomics*, ed. by J. Taylor and M. Woodford, Amsterdam: North- Holland, vol. 1, chap. 26, 1671–1745.

Chen, Y., Y. Chien, and C. Yang (2017): "Aiyagari Meets Ramsey: Optimal Capital Taxation with Incomplete Markets," Working paper, Federal Reserve Bank of St. Louis.

Conesa, J. C., S. Kitao, and D. Krueger (2009): "Taxing Capital? Not a Bad Idea After All!" *American Economic Review*, 99, 25–48.

Dávila, J., J. H. Hong, P. Krusell, and J.-V. Ríos-Rull (2012): "Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks," *Econometrica*, 80, 2431–2467.

Debreu, G. (1970): "Economies with a Finite Set of Equilibria," *Econometrica*, 38, 387–392.

Diamond, P. A. and J. A. Mirrlees (1971): "Optimal Taxation and Public Production I: Production Efficiency," *The American Economic Review*, 61, 8–27.

Dierker, E. and H. Dierker (1972): "The Local Uniqueness of Equilibria," *Econometrica*, 40, 867–881.

Domeij, D. and J. Heathcote (2004): "On The Distributional Effects Of Reducing Capital Taxes," *International Economic Review*, 45, 523–554.

Dyrda, S. and M. Pedroni (2015): "Optimal Fiscal Policy in a Model with Uninsurable Idiosyncratic Shocks," *University of Toronto Working Paper 550*, 1–55.

Gottardi, P., A. Kajii, and T. Nakajima (2015): "Optimal Taxation and Debt with Uninsurable Risks to Human Capital Accumulation," *American Economic Review*, 105, 3443–70.

Heathcote, J. (2005): "Fiscal Policy with Heterogeneous Agents and Incomplete Markets," *The Review of Economic Studies*, 72, 161–188.

Heathcote, J., K. Storesletten, and G. L. Violante (2017): "Optimal Tax Progressivity: An Analytical Framework*," *The Quarterly Journal of Economics*, 132, 1693–1754.

Holter, H. A., D. Krueger, and S. Stepanchuk (2015): "How Do Tax Progressivity and Household Heterogeneity Affect Laffer Curves?" *Working Paper.*

Judd, K. L. (1985): "Redistributive taxation in a simple perfect foresight model," *Journal of Public Economics*, 28, 59–83.

Kindermann, F. and D. Krueger (2015): "High Marginal Tax Rates on the Top 1%? Lessons from a Life Cycle Model with Idiosyncratic Income Risk," .

Le Grand, F. and X. Ragot (2017): "Optimal fiscal policy with heterogeneous agents and aggregate shocks," *Working Paper.*

Ljungqvist, L. and T. J. Sargent (2012): *Recursive Macroeconomic Theory, Third Edition*, MIT Press.

Lucas, R. E. (1990): "Supply-Side Economics: An Analytical Review," *Oxford Economic Papers*, 42, 293–316.

Lucas, R. E. and N. L. Stokey (1983): "Optimal fiscal and monetary policy in an economy without capital," *Journal of Monetary Economics*, 12, 55 – 93.

Marcet, A. and R. Marimon (2011): "Recursive Contracts," .

Park, Y. (2014): "Optimal Taxation in a Limited Commitment Economy," *The Review of Economic Studies*, 81, 884–918.

Piketty, T. and E. Saez (2013): "A Theory of Optimal Inheritance Taxation," *Econometrica*, 81, 1851–1886.

Prescott, E. C. (2004): "Why Do Americans Work So Much More Than Europeans ?" *Federal Reserve Bank of Minneapolis Quarterly Review*, 28, 2–13.

Röhrs, S. and C. Winter (2014): "Reducing Government Debt in the Presence of Inequality," *Unpublished Manuscript*, 49.

Shannon, C. (2006): "A Prevalent Transversality Theorem for Lipschitz Functions," *Proceedings of the American Mathematical Society*, 134.

Straub, L. and I. Werning (2015): "Positive Long Run Capital Taxation: Chamley-Judd Revisited," *NBER Working Paper 20441.*

Tauchen, G. (1986): "Finite State Markov-chain Approximations to Univariate and Vector Autoregressions," *Economic Letters*, 20, 177–181.

Trabandt, M. and H. Uhlig (2011): "The Laffer Curve Revisited," *Journal of Monetary Economics*, 58, 305–327.

APPENDICES

I Derivations

In this section, we provide the derivations of key equations for the household problem, the Ramsey planner's problem and the steady state.

I.1 Households' Problem

A household's labor supply can be expressed as a function of effective wage and consumption, using the first-order condition (FOC) of n_t:

$$
\begin{aligned}
u_c(c_t, n_t)\, e_t \bar{w}_t + u_n(c_t, n_t) = 0 \Rightarrow \\
\frac{-u_n(c_t, n_t)}{u_c(c_t, n_t)} &= e_t \bar{w}_t \Rightarrow \\
\frac{\chi n_t^{\frac{1}{\phi}}}{c_t^{-\sigma}} &= e_t \bar{w}_t \Rightarrow \\
n_t &= \left(\chi^{-1} e_t \bar{w}_t c_t^{-\sigma}\right)^{\phi},
\end{aligned}
$$

and labor income can be also expressed as a function of wage and consumption, as follows:

$$
y_t = \left(\chi^{-1} e_t^{1+\frac{1}{\phi}} \bar{w}_t^{1+\frac{1}{\phi}} c_t^{-\sigma}\right)^{\phi}.
$$

Moreover,

$$
e_t w_t u_{ct} + u_{nt} = 0
$$

will be a useful expression to simplify expressions later. Given the expressions of n_t and y_t, using the FOC w.r.t. a_{t+1} and Kuhn-Tucker condition for the borrowing constraint, a household's policy functions solve the following system of necessary conditions:

$$
\begin{aligned}
u_c(c_t) &\geq \beta(1+\bar{r}_{t+1})\,\mathbb{E}\left[u_c(c_{t+1})\right], \\
0 &= (a_{t+1} + \underline{a})\left(u_c(c_t) - \beta(1+\bar{r}_{t+1})\,\mathbb{E} u_c(c_{t+1})\right), \\
c_t + a_{t+1} &\leq a_t(1+\bar{r}_t) + y_t + T_t \\
a_{t+1} + \underline{a} &\geq 0.
\end{aligned}
$$

I.2 Planner's Problem

Given the planner's problem described in the main text, here we derive the Lagrangian equation (12). First, denote the history of a household's labor productivity from period 0 to t as $h^t = \{h^{t-1}, e_t\}$ where $h^0 = \{e_0\}$. Let θ_{t+1}, η_{t+1} and γ_t represent the present value Lagrangian multipliers for equation (7), (8) and (2) respectively. Then the Lagrangian can be expressed as

$$
\begin{aligned}
L =& \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \Bigg(u\left(c_t\left(h^t\right), n_t\left(h^t\right)\right) \\
& +\theta_{t+1}\left(h^t\right) \left(u_c\left(c_t\left(h^t\right)\right) - \beta\left(1+\bar{r}_{t+1}\right) \sum_{h^{t+1}} \Pi\left(h^{t+1} | h^t\right) u_c\left(c_{t+1}\left(h^{t+1}\right)\right) \right) \\
& -\eta_{t+1}\left(h^t\right)\left(a_{t+1}\left(h^t\right)+\underline{a}\right) \left(u_c\left(c_t\left(h^t\right)\right) - \beta\left(1+\bar{r}_{t+1}\right) \sum_{h^{t+1}} \Pi\left(h^{t+1} | h^t\right) u_c\left(c_{t+1}\left(h^{t+1}\right)\right) \right)\Bigg) \\
& +\sum_{t=0}^{\infty} \beta^t \gamma_t \left(F\left(K_t, N_t\right) + (1-\delta) K_t + B_{t+1} - \left(G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right) \\
=& \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \Bigg(u\left(c_t\left(h^t\right)\right) + u_c\left(c_t\left(h^t\right)\right) \\
& \left(\theta_{t+1}\left(h^t\right) - \theta_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \eta_{t+1}\left(h^t\right)\left(a_{t+1}\left(h^t\right)+\underline{a}\right) + \eta_t\left(h^{t-1}\right)\left(a_t\left(h^{t-1}\right)+\underline{a}\right)\left(1+\bar{r}_t\right) \right)\Bigg) \\
& +\sum_{t=0}^{\infty} \beta^t \gamma_t \left(F\left(K_t, N_t\right) + (1-\delta) K_t + B_{t+1} - \left(G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right).
\end{aligned}
$$

Define $\lambda_{t+1} \equiv \eta_{t+1}\left(a_{t+1}+\underline{a}\right) - \theta_{t+1}$, and the Lagrangian can be further simplified as

$$
\begin{aligned}
L =& \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \Pi\left(h^t\right) \left(u\left(c_t\left(h^t\right)\right) + u_c\left(c_t\left(h^t\right)\right) \left(\lambda_t\left(h^{t-1}\right)\left(1+\bar{r}_t\right) - \lambda_{t+1}\left(h^t\right) \right) \right) \\
& +\sum_{t=0}^{\infty} \beta^t \gamma_t \left(F\left(K_t, N_t\right) + (1-\delta) K_t + B_{t+1} - \left(G_t + T_t + \left(1+\bar{r}_t\right) B_t + \bar{r}_t K_t + \bar{w}_t N_t\right)\right),
\end{aligned}
$$

subject to equation (3), (4), (9), (11) and (10), starting from initial conditions $a_0\left(h^{-1}\right) = a_0, B_0$ and $\lambda_0\left(h^{-1}\right) = 0$.

The first order conditions can be obtained from the Lagrangian, by taking derivatives w.r.t. to the unknowns $\lambda_{t+1}, a_{t+1}, B_{t+1}, T_t, \bar{r}_t, \bar{w}_t$. Then we obtain the set of FOCs in the main text, i.e., equation (13) to (17). The FOCs, together with the constraints, i.e., equation (3), (4), (11) and (10), characterize the necessary conditions for the interior solution of the planner's problem.

In these FOCs, partial derivatives including $\frac{\partial N_t}{\partial T_t}, \frac{\partial N_t}{\partial \bar{r}_t}$ and so on can be expressed as

$$\begin{aligned}
\frac{\partial N_t}{\partial T_t} &= \int \frac{\partial n_t}{\partial c_t}\frac{\partial c_t}{\partial T_t} d\mu_t, \\
\frac{\partial N_t}{\partial \bar{r}_t} &= \int \frac{\partial n_t}{\partial c_t}\frac{\partial c_t}{\partial \bar{r}_t} d\mu_t, \\
\frac{\partial N_t}{\partial \bar{w}_t} &= \int \left(\frac{\partial n_t}{\partial \bar{w}_t} + \frac{\partial n_t}{\partial c_t}\frac{\partial c_t}{\partial \bar{w}_t}\right) d\mu_t.
\end{aligned}$$

Moreover, expressions for $\frac{\partial n_t}{\partial c_t}, \frac{\partial c_t}{\partial T_t}$, and similar partial derivatives are easy to obtain given equation (5), $n_t = \left(\chi^{-1} e_t \bar{w}_t c_t^{-\sigma}\right)^{\phi}$, and (6), $y_t = e_t n_t \bar{w}_t = (e_t \bar{w}_t)^{1+\phi}\left(\chi^{-1} c_t^{-\sigma}\right)^{\phi}$, which describe how n_t and y_t depend on c_t and $\bar{w}_t$. Using also the household budget constraint, equation (3), we obtain the partial derivatives:

$$\begin{aligned}
\frac{\partial n_t}{\partial c_t} &= -\sigma\phi\left(\chi^{-1}\bar{w}_t e_t\right)^{\phi} c_t^{-\sigma\phi-1} = -\sigma\phi\frac{n_t}{c_t}, \\
\frac{\partial n_t}{\partial \bar{w}_t} &= \phi\frac{n_t}{\bar{w}_t}, \\
\frac{\partial c_t}{\partial T_t} &= \frac{\partial y_t}{\partial c_t}\frac{\partial c_t}{\partial T_t} + 1 \quad \Rightarrow \quad \frac{\partial c_t}{\partial T_t} = \frac{1}{1-\frac{\partial y_t}{\partial c_t}} = \frac{1}{1+\sigma\phi\frac{y_t}{c_t}}, \\
\frac{\partial c_t}{\partial \bar{w}_t} &= \frac{\partial y_t}{\partial c_t}\frac{\partial c_t}{\partial \bar{w}_t} + \frac{\partial y_t}{\partial \bar{w}_t} \quad \Rightarrow \quad \frac{\partial c_t}{\partial \bar{w}_t} = \frac{\frac{\partial y_t}{\partial \bar{w}_t}}{1-\frac{\partial y_t}{\partial c_t}} = \frac{e_t n_t}{1+\sigma\phi\frac{y_t}{c_t}},
\end{aligned}$$

I.3 Steady State

Given the assumption that variables are stable at the steady state, we can obtain the FOCs at the steady state by simply replacing variables in the FOCs of the transition dynamics at their steady state values. For example, $\bar{r}_t, \bar{r}_{t+1}$ can be replaced by the steady state value $\bar{r}$. Same for $\bar{w}_t, B_t$, and all the aggregate variables. Notice that households' choice variables a_{t+1}, λ_{t+1} are different, because they are not constant variables but depend on the state of the household. Following Straub and Werning (2015), we focus on the recursive formulation of the problem, that is to say, current period variables a and λ are the state variables which summarize the history and decide next period choice variables a' and λ', together with current period productivity shock e. We can then replace a_t and λ_t with a and λ, and replace a_{t+1} and λ_{t+1} with a' and λ'. Now the steady state solution is characterized by a set of FOCs, as equation (19) to (23), together with following constraints:

$$c + a' = a(1+\bar{r}) + y(e, \bar{w}) + T, \tag{A1}$$

$$G + (1+\bar{r})B + \bar{r}K + \bar{w}N + T \leq F(K, N) + \delta K + B, \tag{A2}$$

$$K = A - B, \tag{A3}$$

$$A = \int a d\mu, \tag{A4}$$

$$N = \int e n d\mu. \tag{A5}$$

II Proofs of Section 3

Proof of Theorem 1

Equation $\gamma = \beta(1 + F_K(K', N') - \delta)\gamma'$ implies in the steady state since $\gamma = \gamma'$ that $1 = \beta(1 + F_K(K, N) - \delta)$, which is the modified golden rule.

Proof of Theorem 2

The idea of the proof is as follows. We first show that the Ramsey problem is generically regular (building on Debreu (1970) and Dierker and Dierker (1972)) which implies that a stationary solution to the the Ramsey problem is locally unique. We then show that the steady state depends continuously on initial conditions such as the initial debt level. Together with the local uniqueness this implies that the steady state does not depend on initial conditions. The proof for now assumes that labor supply is exogenous, $n = N = 1$, and we explain later that the arguments generalize in a straightforward way to the case with endogenous labor supply.

We first show local uniqueness and divide this proof into several steps. As a first step we show that the steady state, which is characterized as a solution to

$$\begin{aligned} u_c(c) &\geq \beta(1+\bar{r}')\mathbb{E}[u_c(c')|e] \text{ with equality if } a' > -\underline{a}, \quad \text{(A6)} \\ u_c(c) - \frac{\partial c}{\partial a'}u_{cc}(c)(\lambda(1+\bar{r}) - \lambda') &= \beta\mathbb{E}_t\left[(1+\bar{r}')u_c(c') + \frac{\partial c'}{\partial a'}u_{cc}(c')(\lambda'(1+\bar{r}') - \lambda'')\right] \end{aligned}$$

$$
\begin{aligned}
& + && \beta\gamma\left(F_K\left(K',N'\right)-\delta-\bar{r}'\right), \\
& && \text{if } a' > -\underline{a}, \text{ otherwise } \lambda' = 0. && \text{(A7)} \\
1 & = && \beta\left(1+F_K\left(K,N\right)-\delta\right), && \text{(A8)} \\
\gamma A & = && \mathbb{E}\left[u_c\left(c\right)\lambda + au_c\left(c\right) + au_{cc}\left(c\right)\left(\lambda\left(1+\bar{r}\right)-\lambda'\right)\right], && \text{(A9)} \\
\gamma N & = && \mathbb{E}\left[enu_c\left(c\right) + enu_{cc}\left(c\right)\left(\lambda\left(1+\bar{r}\right)-\lambda'\right)\right]. && \text{(A10)} \\
c+a' & = && a\left(1+\bar{r}\right)+y\left(e,\bar{w}\right)+T, && \text{(A11)} \\
G+\left(1+\bar{r}\right)B+\bar{r}K+\bar{w}N & \leq && F\left(K,N\right)-\delta K+B, && \text{(A12)} \\
K & = && A-B, && \text{(A13)} \\
A & = && \int ad\mu, && \text{(A14)} \\
N & = && \int end\mu. && \text{(A15)}
\end{aligned}
$$

can be characterized as the solution to two equations $z^{AM}(\bar{r},\bar{w}) = 0$ and $z^{LM}(\bar{r},\bar{w}) = 0$ in the unknowns $\bar{r}$ and $\bar{w}$, the "excess demand" functions in the Asset Market and the Labor Market. Regularity of the steady state then means that these two functions are locally invertible, what we establish in Step 2 below.

Step 1: Characterization of steady state ("excess demand")

To express the steady state as a solution to two equations we first show the existence of steady-state Lagrange Multipliers q.

i) Proof of Existence of steady-state Lagrange Multipliers q

Here we prove the existence and uniqueness of a linear-affine function $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e)$, which solves the steady state equation (A7), which after using (A6), division by γ and defining $q = \lambda/\gamma$ equals
If $a' = -\underline{a}$, $\quad q' = 0$.
If $a' > -\underline{a}$:

If $a' > -\underline{a}$:

$$
\begin{aligned}
& -\frac{\partial c}{\partial a'}u_{cc}\left(c\right)\left[q\left(1+\bar{r}\right)-q'\right] \\
& = \beta\mathbb{E}\left[\frac{\partial c'}{\partial a'}u_{cc}\left(c'\right)\left[q'\left(1+\bar{r}\right)-q''\right]|e\right]+1-\beta\left(1+\bar{r}\right) \\
& = \beta\mathbb{E}\left[-\left(1+\bar{r}\right)\frac{\partial c'}{\partial a''}u_{cc}\left(c'\right)\left[q'\left(1+\bar{r}\right)-q''\right]|e\right]+1-\beta\left(1+\bar{r}\right), \qquad \text{(A16)}
\end{aligned}
$$

where we used that $\frac{\partial c'}{\partial a'} = -(1+\bar{r})\frac{\partial c'}{\partial a''}$.

We establish our results for given interest rate $\bar{r}$ and wage $\bar{w}$, individual saving decisions $a'(a,e)$ and individual consumption decisions $c(a,e)$. Introduce the notation $v := \frac{\partial c}{\partial a'} u_{cc}(c(a,e)) > 0$ and ditto notation for $v' := \frac{\partial c'}{\partial a''} u_{cc}(c(a',e')) > 0$. Rewrite the affine $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e)$ as

$$q'(q,a,e) = \Big[(1+\bar{r})q + H(a,e)/v(a,e)\Big] \cdot K(a,e), \tag{A17}$$

where H, K are nonnegative, $K(a,e) = 0$ for those (a,e) such that $a'(a,e) = -\underline{a}$, so that

$$\alpha_0(a,e) = (1+\bar{r}) \cdot K(a,e) \quad \text{and} \tag{A18}$$

$$\alpha_1(a,e) = K(a,e)H(a,e)/v(a,e). \tag{A19}$$

Similarly

$$q''(q',a',e') = \Big[(1+\bar{r})q' + H'/v'\Big] \cdot K' \tag{A20}$$

for H', K' all nonnegative, $K'(a',e') = 0$ for those (a',e') such that $a''(a',e') = -\underline{a}$.
Insert this into (A16)

$$(1+\bar{r})(K-1)vq + KH - 1 + (1+\bar{r})\beta = (1+\bar{r})\beta\mathbb{E}\left[(1+\bar{r})(K'-1)v'q' + H'K'|e\right] \tag{A21}$$

$$= (1+\bar{r})\beta\mathbb{E}\left[(1+\bar{r})^2(K'-1)v'Kq + (1+\bar{r})(K'-1)HKv'/v + H'K'|e\right] \tag{A22}$$

Gather the "q" terms:

$$(1+\bar{r})(K-1)v = (1+\bar{r})^3\beta\mathbb{E}\left[(K'-1)v'|e\right]K \tag{A23}$$

and solve out for K, taking into account where it must be zero:

$$K = \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1 + (1+\bar{r})^2\beta\mathbb{E}\Big[(1-K')v'/v\Big|e\Big]}. \tag{A24}$$

We define an iteration of functions which converge to the solution. As initialization we set

$K_0(a,e) \equiv 0$ and define inductively

$$K_{n+1}(a,e) = \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1+(1+\bar{r})^2\beta\mathbb{E}\Big[\big\{1-K_n(a'(a,e),e')\big\}v'(a',e')/v(a,e)\Big|e\Big]} \tag{A25}$$

By induction, it follows that $1 \geq K_{m+1} \geq K_m \geq \dots K_0 = 0$. This is obviously true for $n = 0$. For $m+1$ it follows from $K_{m+1} \geq K_m$ that

$$\begin{aligned} K_{m+2}(a,e) &= \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1+(1+\bar{r})^2\beta\mathbb{E}\Big[\big\{1-K_{m+1}(a'(a,e),e')\big\}v'(a',e')/v(a,e)\Big|e\Big]} && \text{(A26)} \\ &\geq \frac{1_{\{(a,e);a'>-\underline{a}\}}}{1+(1+\bar{r})^2\beta\mathbb{E}\Big[\big\{1-K_m(a'(a,e),e')\big\}v'(a',e')/v(a,e)\Big|e\Big]} && \text{(A27)} \\ &= K_{m+1}(a,e). && \text{(A28)} \end{aligned}$$

We therefore obtain a well-defined measurable function K defined by the pointwise $K(a,e) := \sup_m K_m(a,e)$ ($\in [0,1]$ and 0 when $a' = -\underline{a}$).
That was the "q" terms. For the constant term:

$$KH - 1 + (1+\bar{r})\beta = (1+\bar{r})\beta\mathbb{E}\Big[(1+\bar{r})(K'-1)HKv'/v + H'K'\Big|e\Big], \quad \text{i.e.} \tag{A29}$$

$$H \cdot \underbrace{\Big\{1+(1+\bar{r})^2\beta\mathbb{E}\Big[(1-K')v'/v\Big|e\Big]\Big\} \cdot K}_{=1 \text{ on } \{(a,e);a'>-\underline{a}\} \text{ by (A24)}} = 1-(1+\bar{r})\beta + (1+\bar{r})\beta\mathbb{E}\Big[H'K'\Big|e\Big] \tag{A30}$$

As we can safely put $H = 0$ on $\{(a,e); a' = -\underline{a}\}$, we can iterate from $H_0(a,e) \equiv 0$ the relation

$$H_{m+1}(a,e) = \Big\{1-(1+\bar{r})\beta + (1+\bar{r})\beta\mathbb{E}\Big[H_m(a'(a,e),e')K'(a'(a,e),e')\Big|e\Big]\Big\} \cdot 1_{\{(a,e);a'>-\underline{a}\}} \tag{A31}$$

Now the condition $(1+\bar{r})\beta \sup_e \mathbb{E}K'(a'(a,e),e') < 1$ (recall that $K' \in [0,1]$ and $(1+\bar{r})\beta < 1$ – and except in the trivial case, zero when the credit constraint is binding) is sufficient for a contraction and unique solution H; if we start at 0, then we have bounded monotonicity $1 \geq H_{m+1} \geq H_m \geq 0$, and thus H defined by $H(a,e) := \sup_m H_m(a,e) \in [0,1]$ does the job. We have therefore established the existence a solution $q'(q,a,e) = \alpha_0(a,e)\,q + \alpha_1(a,e) = \Big[(1+\bar{r})q + H(a,e)/v(a,e)\Big] \cdot K(a,e)$.

ii) "Excess Demand" Functions

For a given $\bar{w}$, equations (A6) and (A11) describe households consumption and savings behavior as a function of $\bar{r}$, resulting in an aggregate asset supply function $S(\bar{r}, \bar{w})$.[19]

Asset demand D, the sum of capital and bonds, follows from the government budget constraint (A12) using (A13) and (A15)

$$D(\bar{r}, \bar{w}) := A = K + B = \frac{F(K, N) - \delta K - \bar{w}N - G}{\bar{r}},$$

which, since we already established that capital K satisfies the modified golden rule (equation (A8)), is actually just describing how many government bonds are demanded. We therefore define

$$z^{AM}(\bar{r}, \bar{w}) = D(\bar{r}, \bar{w}) - S(\bar{r}, \bar{w}).$$

A solution $\bar{r}$ (for given $\bar{w}$) to $z^{AM}(\bar{r}, \bar{w}) = 0$ fully characterizes a stationary Aiyagari economy (and solves equation (A14)).

To derive the second equation $z^{LM}(\bar{r}, \bar{w})$ we use the remaining equations (A9) and (A10). After division by γ equation (A9) reads

$$A = \mathbb{E}\left[u_c(c)\, q + a\frac{u_c(c)}{\gamma} + a u_{cc}(c)\left(q(1+\bar{r}) - q'\right)\right],$$

where q' depends on $\bar{r}$, $\bar{w}$ and other parameters. Solving this equation for γ yields a function $\tilde{\gamma}(\cdot)$:

$$\tilde{\gamma}(\cdot) = \frac{\mathbb{E}\left[\frac{a}{A}u_c(c)\right]}{1 - \mathbb{E}\left[\frac{u_c(c)q}{A} + a u_{cc}(c)\left(q(1+\bar{r}) - q'\right)\right]}$$

Plugging this function into (A10) (and noting $n = 1$) yields

$$\int e n d\mu = \mathbb{E}\left[e[\frac{u_c(c)}{\tilde{\gamma}} + u_{cc}(c)\left(q(1+\bar{r}) - q'\right)]\right].$$

[19] While it is conceivable that aggregate asset supply is not unique given $\bar{r}$ and $\bar{w}$, this is not a concern here since we impose the standard assumption that the planner picks the unique welfare maximizing allocation.

We therefore define

$$z^{LM}(\bar{r},\bar{w}) := \mathbb{E}\left[e[\frac{u_c(c)}{\tilde{\gamma}} + u_{cc}(c)\left(q\left(1+\bar{r}\right)-q'\right)]\right].$$

The optimal steady state then satisfies

$$z^{LM}(\bar{r},\bar{w}) = \int end\mu.$$

<u>Step 2: Local Invertibility</u>

We first show that the interest rate $\bar{r}$ can generically (in the sense of Debreu (1970)) be expressed locally as a function of $\bar{w}$ (and other parameters). This first step is what is not feasible in complete markets models and is thus the reason why we obtain local uniqueness here but not in complete markets models. After that we show that $\bar{w}$ is also generically locally invertible. This second step holds both in complete and incomplete markets models.

<u>i) Interest Rate</u>

Acemoglu and Jensen (2015) show that a tightening of the borrowing limit leads to an increase in the supply of assets for given $\bar{r}$ and $\bar{w}$ but will not change the modified golden rule level of capital.[20] The transversality theorem (see e.g. Dierker and Dierker (1972), Shannon (2006)) implies then that

$$\frac{\partial z^{AM}(\bar{r},\bar{w})}{\partial \bar{r}} \neq 0,$$

which implies that $\bar{r}$ is locally invertible and is thus a function of $\bar{w}$ and can be written as $\bar{r}(\bar{w})$. The transversality therorem allows us to not directly compute the derivative with respect to $\bar{r}$ but instead to consider the derivative for some parameter, the exogenous borrowing constraint, and then infer the local invertibility for $\bar{r}$. Clearly, this line of arguments does not apply in complete markets models, which do not have a steady/state asset demand function but only a correspondence, and where the arguments of Acemoglu and Jensen (2015) are not applicable since tighenting the borrowing constraint is not a well-defined experiment in the standard complete markets model.

[20] Acemoglu and Jensen (2015) call such an experiment a positive shock. Their objective is more demanding then just showing an increase in the supply function. They characterize the response of the equilibrium output per capita which has to take into account the endogeneity of prices.

ii) Wage The after tax wage $\bar{w}$ is determined as the solution to

$$z^{LM}(\bar{r}(\bar{w}), \bar{w}, \mu) = \sum_{e \in E} e\mu(e),$$

where we have plugged in $\bar{r}(\bar{w})$ and use, consistent with the numerical implementation, a more convenient discrete space $E = \{e_1 < e_2, \ldots < e_N\}$.
We now follow Debreu (1970) and apply Sard's theorem to the function $F : \mathbb{R}^{N+1} \to \mathbb{R}^N$,

$$F(\bar{w}, \{\mu(e_i)\}_{i=1}^N) = (\mu(e_1), \mu(e_2), \ldots, \mu(e_{N-1}), \frac{z^{LM}(\bar{r}(\bar{w}), \bar{w}, \mu) - \sum_{i=1}^{N-1} e_i \mu(e_i)}{e_N}).$$

The optimal solution is characterized as $F(\bar{w}, \{\mu(e_i)\}_{i=1}^N) = (\mu(e_1), \ldots, \mu(e_N))$ and Sard's theorem, which implies that the set of critical values has measure zero, delivers the local invertibility result.[21]

We could also again apply the transversality theorem (which is based on Sard's theorem) and perturb the highly-dimensional whole distribution to show that the derivative with respect to the one-dimensional variable $\bar{w}$ is generically non-zero. Again, the transversality theorem implies that we do not have to calculate the derivative w.r.t. $\bar{w}$ but that instead it is sufficient to show that not for all all $\mu(e_n)$ the derivative of the last element of F is equal to zero. As this case- a function or integral which evaluates to zero at every point would be zero, such that all expectations are zero implying that the resource constraint is not binding, a contradiction - we again obtain local invertibility.

Both the distribution μ and the after-tax wage $\bar{w}$ live on a compact space K, $(\bar{w}, \{\mu(e_i)\}_{i=1}^N) \in K$. This is obvious for $\mu(e_i) \in [0, 1]$ and for $\bar{w}$ follows from Aiyagari (1994b) who ensures that no-one is willing to work in the market at a wage of 0 and the marginal productivity of labor is bounded since capital and hours (time) are.

The inverse image $F^{-1}(\mu(e_1), \ldots, \mu(e_N))$ of a regular value $((\mu(e_1), \ldots, \mu(e_N))$ is compact since F is continuous and K is compact. Consider now $e := (\bar{w}, \{\mu(e_i)\}_{i=1}^N) \in F^{-1}(\mu(e_1), \ldots, \mu(e_N))$, for a regular $((\mu(e_1), \ldots, \mu(e_N))$ implying that the Jacobian does not vanish. The inverse function theorem implies that for each such e there is an open neighborhood U_e of e such that $F^{-1}(\mu(e_1), \ldots, \mu(e_N)) \cap U_e = \{e\}$. Since $F^{-1}(\mu(e_1), \ldots, \mu(e_N))$ is compact it can be

[21] For a continuously differentiable function $F : U \subset \mathbb{R}^m \to \mathbb{R}^n$, a point $e \in U$ is a critical point if the Jacobian matrix of F at e has rank smaller than n. A point $\mu \in \mathbb{R}^n$ is a critical value if there is a critical point $e \in U$ such that $F(e) = \mu$. A point $\mu \in \mathbb{R}^n$ is a regular value if it is not a critical value.

covered by finite number of open sets U_e and therefore is finite.

This implies that the set of $\mu(e_1), \ldots, \mu(e_N)$ for which an infinite number of steady states exists consists of critical values only and has therefore measure zero (and so does its closure). Vice versa, the number of steady state solution is generically, that is on a measure one, finite.

Step 3: Continuity w.r.t. Initial Conditions

Berge's maximum theorem (see Theorem 17.31 in Aliprantis and Border (2006) for the infinite-dimensional version) implies that the optimal policy path, and thus in particular the steady state, depends continuously on the initial level of government debt. Here we use the same topology, the product topology, as in Appendix A of Aiyagari (1994a). The equations describing the constraints of the Ramsey planner problem are continuous and the constrained set is by Tychonoff's theorem compact. The maximand of the Ramsey problem is continuous as well. These properties imply that a solution to the optimal tax problem exists, as shown in Aiyagari (1994a), and allow us to apply Berge's maximum theorem. Finally, note that a function is continuous in the product topology iff all its projections are continuous, implying that the usual real analysis ϵ/δ characterization of continuity holds for all t and in particular for arbitrarily large t.

Therefore, the function $\zeta : \mathbb{R} \to \mathbb{R}^n$ mapping the initial government debt level into the steady state policies (tax rates and debt level) is continuous.

Step 4: Independence of Initial Conditions

We have shown in Step 2 that the set of solutions to the first order conditions is finite. These first-order conditions do not depend on the initial level of government debt. That is the finite set of solutions does not depend on the initial level of debt. The first-order conditions are necessary conditions for an optimum, implying that every optimal policy has to be one of the finite solutions to the first-order conditions. What remains to be shown is that each initial debt levels always yields the same solution to the first-order conditions, that is that there is no selection of these solutions based on initial conditions. Using our results above, this is straightforward.

A continuous function mapping into a discrete set is constant, implying that ζ maps every initial debt level to the same steady-state policy.

Remarks:

Elastic Labor supply

The same arguments hold when labor supply is elastic. We then define

$$z^{LM}(\bar{r},\bar{w}) := \mathbb{E}\left[en[\frac{u_c\left(c\right)}{\tilde{\gamma}} + u_{cc}\left(c\right)\left(q\left(1+\bar{r}\right)-q'\right)]\right] + \left(F_N\left(K,N\right)-\bar{w}\right)\frac{\partial N}{\partial \bar{w}}$$

and the optimal steady state then satisfies

$$z^{LM}(\bar{r},\bar{w}) = \sum_e \pi_e en\left(e,\bar{w}\right).$$

<u>Additional Policy Instrument: Transfers</u>

The same arguments hold when the government can use lump-sum transfers as an additional instrument. We then define a function

$$z^{T}(\bar{r},\bar{w},T) := \mathbb{E}\left[\frac{u_c\left(c\right)}{\tilde{\gamma}} + u_{cc}\left(c\right)\left(q\left(1+\bar{r}\right)-q'\right)]\right] + \left(F_N\left(K,N\right)-\bar{w}\right)\frac{\partial N}{\partial T} - 1,$$

so that the first-order condition reads as

$$z^{T}(\bar{r},\bar{w},T) = 0.$$

The arguments as above made for F now apply to the function $\tilde{F}$:

$$\begin{aligned}&\tilde{F}(\bar{w},T,\{\mu(e_i)\}_{i=1}^{N})\\ =&(\mu(e_1),\mu(e_2),\ldots,\mu(e_{N-1}) + z^{T}(\bar{r}(\bar{w},T),\bar{w},T), \frac{z^{LM}(\bar{r}(\bar{w},T),\bar{w},T,\mu) - \sum_{i=1}^{N-1} e_i\mu(e_i)}{e_N}).\end{aligned}$$

The optimal solution is characterized as $\tilde{F}(\bar{w},T,\{\mu(e_i)\}_{i=1}^{N}) = (\mu(e_1),\ldots,\mu(e_N))$.

III Computational Algorithms

III.1 Steady State

To numerically compute the steady state, we first need to introduce the steady state distribution of state variables (a,λ,e), represented by a density function $p\left(a,\lambda,e\right)$. Moreover, we denote the density function of (a,e) as $m\left(a,e\right)$. We discretize e, using the method of Tauchen (1986) and 7 equally spaced values for e in $E=[-3\sigma_e,3\sigma_e]$. Now the steady state equations involving expectation and integration can be explicitly expressed using p and m. Equation (24), (22), (23), (A4) and (A5) are now:

$$\begin{aligned}
\gamma &= \sum_e \int \int \left(u_c(c) + \frac{\partial c}{\partial T} u_{cc}(c) \left[\lambda (1+\bar{r}) - \lambda' \right] \right) p(a, \lambda, e)\, da d\lambda \\
&\quad + \gamma \left(F_N(K, N) - \bar{w} \right) \frac{\partial N}{\partial T}, && \text{(A32)} \\
\gamma A &= \sum_e \int \int u_c(c) \lambda p(a, \lambda, e)\, da d\lambda \\
&\quad + \gamma \left(F_N(K, N) - \bar{w} \right) \frac{\partial N}{\partial \bar{r}} \\
&\quad + \sum_e \int \int a \left(u_c(c) + u_{cc}(c) \left[\lambda (1+\bar{r}) - \lambda' \right] \right) p(a, \lambda, e)\, da d\lambda, && \text{(A33)} \\
\gamma N &= \gamma \left(F_N(K, N) - \bar{w} \right) \frac{\partial N}{\partial \bar{w}} \\
&\quad + \sum_e \int \int en(u_c(c) + u_{cc}(c) \left(\lambda (1+\bar{r}) - \lambda' \right)) p(a, \lambda, e)\, da d\lambda, && \text{(A34)} \\
N &= \sum_e \int enm(a, e)\, da, && \text{(A35)} \\
A &= \sum_e \int am(a, e)\, da. && \text{(A36)}
\end{aligned}$$

Moreover, the density functions satisfy

$$\begin{aligned}
p(a', \lambda', e') &= \sum_e \pi_{ee'} \int I \left[g_{a'}(a, e) = a', g_{\lambda'}(a, \lambda, e) = \lambda' \right] p(a, \lambda, e)\, da d\lambda, && \text{(A37)} \\
m(a', e') &= \sum_e \pi_{ee'} \int I \left[g_{a'}(a, e) = a' \right] m(a, e)\, da. && \text{(A38)}
\end{aligned}$$

Using the steady state equations, i.e., equation 19 to 21, A1, A2, and A32 to A38, we compute the steady state variables according to the below steps.

1. Guess T.
2. Guess $\bar{w}$. Solve for $\bar{r}(\bar{w})$ following Aiyagari (1995):
    (a) Solve for K from (21).

(b) Guess $\bar{r}$ and solve the household's problem: solve for $c(a,e), a'(a,e)$ from (19) and (A1), keeping in mind that $n = \left(\chi^{-1}\bar{w}ec^{-\sigma}\right)^{\frac{1}{\phi}}, y = \left(\chi^{-1}\bar{w}^{1+\phi}e^{1+\phi}c^{-\sigma}\right)^{\frac{1}{\phi}}$. To solve for $c(a,e), a'(a,e)$ we use an endogenous grid apporach, as described by Carrol (2006).

(c) Compute N from (A35).

(d) Solve for $m(a,e)$ or equivalently $\mu(.,e)$ from (A38).

(e) Solve for A from (A36).

(f) Solve for B from (A3).

(g) Verify $\bar{r}$ using (A2). If the equation is not satisfied, update $\bar{r}$.

3. Define $q \equiv \frac{\lambda}{\gamma}$, and solve for $q'(a,q,e)$ by iterating on $q'(a,q,e)$ using (20) until q' converges. Guess $q'(a,q,e) = g^0_{q'}(a,q,e)$, and then use equation (20) to find the new $q'(a,q,e) = g^1_{q'}(a,q,e)$ as follows:

$$q' = \frac{-\frac{\partial c}{\partial a'}u_{cc}(c)\,q(1+\bar{r}) + \beta\mathbb{E}\left[\frac{\partial c'}{\partial a'}u_{cc}(c')\,q''|e\right] - 1 + \beta(1+\bar{r})}{-\frac{\partial c}{\partial a'}u_{cc}(c) + \beta\mathbb{E}\left[\frac{\partial c'}{\partial a'}u_{cc}(c')(1+\bar{r})\right]}$$

where $\mathbb{E}\left[\frac{\partial c'}{\partial a'}u_{cc}(c')\,q''|e\right]$ can be computed using $g^0_{q'}(a,q,e)$, and the new q' gives us the new policy function, denoted as $g^1_{q'}(a,q,e)$. Keep updating until $g^i_{q'}(a,q,e)$ converges to $g_{q'}(a,q,e)$. It can be proven that the above functional equation is a contraction mapping.

4. Solve for γ from (22)

5. Check whether (23) is satisfied. If so, stop. Otherwise update $\bar{w}$.

6. Check whether (A32) is satisfied. If so, stop. Otherwise update T.

III.2 Transition

Below we outline the algorithm for computing the transition from the model calibrated to the U.S. economy to the optimal long run steady state.:

1. Choose a number of transition periods, J.

2. Compute the optimal long run steady state as outlined in III.1 and obtain $a_{t+1}(a_t, e_t)$, $c_t(a_t, e_t)$ at time J.

3. Compute the steady state for the economy calibrated to the U.S. and obtain $m_0(a_0, e_0)$ A_0, B_0, K_0.

4. Guess $\{\bar{w}_t, \bar{r}_t\}_{t=0}^{J}$.

5. Solve households' problems by backward induction and obtain $a_{t+1}(a_t, e_t)$, $c_t(a_t, e_t)$.

6. Compute distribution of asset and productivity $m_t(a_t, e_t)$, using simulation starting from $m_0(a_0, e_0)$.

7. Compute A_t and N_t from (9) and (10).

8. Compute K_t and B_{t+1} going backwards using (11) and (2), namely,

$$\begin{aligned} K_t &= A_t - B_t \\ B_{t+1} &= G_t + (1 + \bar{r}_t) B_t + \bar{r}_t K_t + \bar{w}_t N_t + T_t - F(K_t, N_t) + \delta K. \end{aligned}$$

9. Compute γ_t backward using

$$\gamma_t = \beta (1 + F_K(K_{t+1}, N_{t+1}) - \delta) \gamma_{t+1}$$

10. Solve for $\lambda_{t+1}(a_t, \lambda_t, e_t)$ from 14.

11. Compute p_t forward by simulations using p_0 and the policy functions: $a_{t+1}(a_t, e_t)$, $c_t(a_t, e_t)$ and $\lambda_{t+1}(a_t, \lambda_t, e_t)$.

12. Check the errors implied by the guessed $\{\bar{w}_t, \bar{r}_t\}_{t=0}^{J}$. This means check the equations 16, and 17. If they are not satisfied, update the guess for $\{\bar{w}_t, \bar{r}_t\}_{t=0}^{J}$. In practice we do this by a minimization routine, which minimizes the sum of the squared errors in the equations.