Managing Inequality over the Business Cycles: Optimal Policies with Heterogeneous Agents and Aggregate Shocks[*]

François Le Grand   Xavier Ragot[†]

Preliminary and incomplete

Abstract

We solve for optimal Ramsey policies in heterogeneous-agent models with aggregate shocks. We provide a simple theory based on projection on the space of idiosyncratic histories, to present a finite-dimensional state-space heterogeneous agent models. This improves current algorithms for solving such models with aggregate shocks, and allows simulating easily optimal policies. We apply this methodology to the optimal provision of a public good over the business cycle, when households face time-varying unemployment risk.

Keywords: Incomplete markets, optimal policies, heterogeneous agent models.
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1 Introduction

Incomplete insurance-market economies provide a useful framework for examining many relevant aspects of inequalities, among households, firms or even countries. In these models, infinitely-lived agents face incomplete insurance markets and borrowing limits that prevent them from perfectly hedging their idiosyncratic risk, in line with the Bewley-Huggett-Aiyagari literature (Bewley 1983, Imrohoroğlu 1989, Huggett 1993, Aiyagari 1994, Krusell and Smith 1998). These frameworks are becoming increasingly popular, since they fill a gap between micro- and macroeconomics and enable the inclusion of aggregate shocks and a number of additional frictions on both the goods and labor markets. However, in terms of normative analysis, little is known about optimal policies in these environments, due to the difficulties generated by the large and time-varying heterogeneity across agents. This is unfortunate, since a vast literature, reviewed below, suggests that the interaction between wealth inequalities and capital accumulation has first-order implications for the design of optimal time-varying policies.

This paper presents a methodological contribution that offers a general and tractable representation of incomplete insurance-market economies. This representation allows us to easily solve the Ramsey problem in economies with both capital and aggregate shocks. We apply our framework to the optimal provision of a public good over the business cycle as a simple application of the methodology.

Heterogeneity indeed increases with time in incomplete insurance-market economies because agents differ according to the full history of their idiosyncratic risk realizations. Huggett (1993), using the results of Hopenhayn and Prescott (1992), and Aiyagari (1994) have shown that economies without aggregate risk have a recursive structure when the distribution of wealth is introduced as a state variable. Unfortunately, the distribution of wealth has infinite support, which is at the root of many analytical difficulties. Our methodological contribution is to represent incomplete insurance-market economies as economies with finite support.

The basic idea is first to go back in the sequential representation to consider the set of idiosyncratic histories at each period. Then one can use a time-invariant partition of these histories $\mathcal{P}$ such that each agent, at each period, belongs to one and exactly one element of this partition $p \in \mathcal{P}$. Then we show that there is a simple way to aggregate heterogeneity within each element p, such that one can follow the dynamics of the finite number of elements $p \in \mathcal{P}$, instead of the whole distribution.

Our approach is thus to project the model on the space of idiosyncratic histories, to simulate

the model and derive optimal policies. What is the proper choice of the partition? Some explicit partitions can be constructed, based on a truncation of idiosyncratic histories (as we did in a previous version of this paper see LeGrand and Ragot (2017)). Each agent having the same history of the idiosyncratic shock for the last N periods are in the same element of $\mathcal{P}$ (for a given length N). Partitions can be defined more generally using information about the steady-state equilibrium to gather histories in an efficient manner. The interest of this construction is threefold. First, constructing the projection, we show that it provides some improvement on previous algorithms to solve incomplete-insurance market with aggregate shocks, as Reiter (2009). In particular, we use the information about the steady-state distribution of wealth and we capture the heterogeneity in Euler equations within each elements of the partition $\mathcal{P}$. Second and more importantly, this construction allows us to derive optimal Ramsey program with aggregate shocks. The basic idea is as follows. One can use tools developed in dynamic contracts, namely Marcet and Marimon (2011) applied on elements of the partition $p \in \mathcal{P}$, to derive first-order conditions for the planner. These conditions are then easy to simulate with aggregate shocks. In these economies, the difficult part is to find the steady-state of the optimal Ramsey policies (and check that this interior solution is consistent with second-order conditions). The projection techniques provides then a simple algorithm using information from a general Bewley model, to show the convergence of the instrument of the planner. We apply this methodology to the question of the optimal provision of public good in an economy with uninsurable employment risk and aggregate shocks, when the public good is financed by a tax on labor. In the quantitative application, we use the calibration of Krueger, Mitman, and Perri (2017), which reproduces a realistic time-varying unemployment risk, income risk and wealth distribution. This example is purposely a simple normative question, in order to present the methodology in a transparent way. A third interest of this construction is to provide a theoretical representation of algorithms using projections methods. The gain is that equations, such as the first-order conditions of the planner, are easy to understand economically. This help us identify propagation channels in these very complex models.

This paper is mainly related to two strands of the literature. The first one is the computation of incomplete insurance markets with aggregate shocks. After the seminal paper of Krusell and Smith (1998), incomplete insurance market models with aggregate shocks have first been solved using a fixed point on simple expectations rules. Since, the work of Reiter (2009), the literature has used projection methods to first simplify the distribution of wealth and then simulate the model. These techniques are now used in various setups, to solve discrete-time models Winberry

(2016) or models first written in continuous time as in Ahn, Kaplan, Moll, Winberry, and Wolf (2017). In this literature, our contribution is to improve on simple projections methods by using more information about the steady-state Bewley model.

Second, this paper is related to the literature on optimal (Ramsey) policies in heterogeneous agent models. This literature is thin and very recent. First, Açikgöz (2015) provides an algorithm to solve for the steady-state allocation of the Ramsey program, based on assumptions on functional form. Nuño and Moll (2017) use a continuous-time approach without aggregate shock and rely on projection methods to determine the steady-state allocation. Bhandari, Evans, Golosov, and Sargent (2016) present a solution method of models with aggregate shocks, which relies on perturbation methods around time-varying allocations. They solve the model approximating the distribution by 100,000 agents. Compared to these models our contribution is to provide a general representation allowing to simulate general models with optimal policies and aggregate shocks.

The rest of the paper is organized as follows. Section 2 presents the simple environment, on which our methodology will be applied. Section 3 presents the projection in the space of idiosyncratic histories in the general case. Section 4 presents solution techniques to derive optimal policies. Section 5 analyses in more detail how Reiter (2009) can be understood as an implicit partition to provide some improvement on its algorithm. Section 6 provides two numerical examples, a first one without optimal policies to benchmark our method with other ones presented in the literature. The second one computes optimal time-varying fiscal policy.

2   The economy

We consider a discrete-time setup. The economy features a single good and is populated by a population of size 1 of agents distributed on a segment I according to a measure $\ell(\cdot)$. We assume that the law of large number holds.

2.1   Preferences

Agents derive utility in each period from private consumption c and from the provision of a public good G. The period utility function is denoted $U(c,G)$ and is assumed to be separable in private consumption and public good provision. Its functional form is

$$U(c,G) = u(c) + v(G),$$

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where u and v are twice continuously derivable functions from $\mathbb{R}_+$ onto $\mathbb{R}$. Functions u and v are strictly increasing and concave, with $\lim_{c \to 0+} u'(c) = \infty$.

In what follows, we use a CRRA utility function:

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma} + \chi \frac{G^{1-\gamma_G} - 1}{1 - \gamma_G}, \tag{1}$$

where $0 < \gamma, \gamma_G \neq 1$. When $\gamma = \gamma_G = 1$, the utility function is simply $U(c, G) = \log(c) + \chi \log(G)$ (the two other cases $\gamma = 1 \neq \gamma_G$ and $\gamma \neq 1 = \gamma_G$ are straightforward to deduce).

Agents have standard additive intertemporal preferences, with a constant discount factor $\beta > 0$. They therefore rank consumption and public good streams, denoted respectively by $(c_t)_{t \geq 0}$ and $(G_t)_{t \geq 0}$, using the intertemporal criterion $\sum_{t=0}^{\infty} \beta^t U(c_t, G_t)$.

2.2 Risks

We consider a general setup where agents face both aggregate risk, time-varying unemployment risk, and idiosyncratic productivity risk, as modeled by Krueger, Mitman, and Perri (2017). As will be clear in the quantitative analysis below, this general setup allows us to match the wealth distribution and a realistic dynamics of the labor market.

Aggregate risk. The aggregate risk will affect both aggregate productivity and the unemployment risk. Formally, the aggregate risk is represented by a probability space $(\mathcal{Z}^{\infty}, \mathcal{F}, \mathbb{P})$. At a given date t, the aggregate state is denoted s_t and takes values in the state space $\mathcal{Z} \subset \mathbb{R}^+$. We assume the aggregate risk to be a Markov process.[1] The history of aggregate shocks up to time t is denoted $z^t = \{z_0, \ldots, z_t\} \in \mathcal{Z}^{t+1}$. Finally, the period-0 probability density function of any history z^t is denoted $m^t(z^t)$.

For the sake of clarity, for any random variable $X_t : \mathcal{Z}^t \to \mathbb{R}$, we will denote X_t, instead of $X_t(z^t)$, its realization in state z^t,

Employment risk. At the beginning of each period, agents face an uninsurable idiosyncratic employment risk, denoted e_t at date t. The employment status e_t can take two values, e and u, corresponding to employment and unemployment respectively. We denote by $\mathcal{E} = \{e, u\}$ the set of possible employment status. Employed agents with $e_t = e$ can supply inelastically one unit of labor, and they earn a before-tax real wage, denoted w_t at date t. Unemployed agents with $e_t = u$ cannot work and will receive unemployment benefits financed by social

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[1]In the quantitative part, we will assume that it more specifically follows an AR(1) process.

contributions, that we describe further below. A history of idiosyncratic shocks up to date t is denoted $e^t = \{e_0, \ldots, e_t\} \in \{0, 1\}^{t+1}$.

The employment status e_t follows a discrete Markov process with transition matrix $M_t(z^t) \in [0,1]^{2 \times 2}$ that is assumed to depend on the history of aggregate shocks up to date t. The job separation rate between periods $t-1$ and t is denoted $l_t(z^t)$, while $f_t(z^t)$ is the job finding rate between $t-1$ and t. The time-varying transition matrix across employment status is therefore:

$$M_t(z^t) = \begin{bmatrix} 1 - f_t(z^t) & f_t(z^t) \\ l_t(z^t) & 1 - l_t(z^t) \end{bmatrix}. \tag{2}$$

As in Krusell and Smith (1998) and Krueger, Mitman, and Perri (2017), we assume that the share of the population that unemployed only depends on the current *aggregate* state, and that transition probabilities s and f actually only on the current and past aggregate states. We denote by $\eta_{u,t}(z_t)$ and $\eta_{e,t}(z_t)$ the populations of unemployed and employed agents respectively – where $\eta_{u,t}(z_t) + \eta_{e,t}(z_t) = 1$ at all dates.

Productivity risk. The individual productivity of agents is stochastic. At any date t, the individual productivity status is denoted y_t and takes values in a finite set $\mathcal{Y} \subset \mathbb{R}_+$. The cardinality of the set $\mathcal{Y}$ is denoted $Card\ \mathcal{Y}$ and is thus the number of different idiosyncratic productivity levels. Large values of y_t correspond to highly productive agents. The before-tax wage earned by an employed agent will be the product of an aggregate wage denoted w_t, depending on aggregate shock, and of the individual productivity y: It is $y_t w_t$ An unemployed agent will also carry an idiosyncratic productivity level that will affect her unemployment benefits.

The history of productivity shocks of a given agent up to date t is denoted $y^t = \{y_0, \ldots, y_t\}$. The productivity status follows a first-order Markov process where the transition probability from state $y_{t-1} = y$ to $y_t = y'$ is constant and denoted $\pi_{yy'}$. In particular, it is independent of the employment status of the agent. We denote by η_y the share of agents endowed with individual productivity level y. This share is constant through time because of assumptions on transition probabilities $\pi_{yy'}$.

The individual status of each agent is characterized by her employment status e and personal productivity level y. At any date t, we will denote by $s_t = (e_t, y_t)$ the date-t individual status of any agent. The set of possible individual status is denoted $\mathcal{S} = \mathcal{E} \times \mathcal{Y}$. Finally, we denote as s^t a history until period t: $s^t = \{\ldots, s_{t-1}, s_t\}$.

2.3 Production

The good is produced by a unique profit-maximizing representative firm. This firm is endowed with a production technology that transforms, at date t, labor L_t and capital K_{t-1} into Y_t output units of the single good. The production function is a Cobb-Douglas function with parameter $\alpha \in (0,1)$ featuring constant returns-to-scale. The capital must be installed one period before production and the total productivity factor Z_t is stochastic. Denoting as $\delta > 0$ the constant capital depreciation, the output Y_t is formally defined as follows:

$$Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1}. \tag{3}$$

where L_t is the labor supply expressed in efficient units. We have:

$$L_t = \eta_{e,t}(z) \sum_{y \in \mathcal{Y}} \eta_y y.$$

The total productivity factor is simply the exponential of the aggregate shock z_t:

$$Z_t = \exp(z_t). \tag{4}$$

The two factor prices at date t are the aggregate before-tax wage rate w_t and the capital return r_t. As we explain further below, we assume that while labor is taxed at a linear rate, capital is not taxed. The profit maximization of the producing firm implies the following factor prices.

$$w_t = (1 - \alpha) Z_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha}, \tag{5}$$

$$r_t = \alpha Z_t \left(\frac{K_{t-1}}{L_t} \right)^{\alpha-1} - \delta. \tag{6}$$

2.4 Social contributions and taxes

The government raises both social contributions and labor taxes, which have two distinct objectives. Social contributions solely serve to finance unemployment benefits, while labor tax serves to finance the public good. We have chosen the above setup, since it features one of the simplest fiscal system we can think of. It will simplify the comparison between complete and incomplete-market economies. Indeed, the labor tax is non-distorting as the labor supply is inelastic. As a consequence, the complete market allocation reproduces the first-best allocation. The difference between complete and incomplete market economies will only result from the distributional effect of labor tax.

Unemployment insurance. Unemployed agents receive at any date an unemployment benefit that is equal to a constant fraction of the wage the agent would earn if she were employed. The replacement rate, denoted ϕ, being constant, the unemployment benefit of an agent endowed with productivity y equals $\phi w_t y$.

The unemployment benefits are financed by social contributions. These contributions are paid *by all* agents, no matter whether they are employed or unemployed. Contributions amount to a constant proportion $\tau_t(z)$ of the wage and this proportion is identical for all agents, but depends on the current aggregate state z. The contribution τ_t is set such that the unemployment insurance (UI) scheme is balanced at any date t, since we rule out the possibility of social debt. The balance budget of the unemployment insurance scheme can be expressed as:

$$\eta_{u,t}(z) \sum_{y \in \mathcal{Y}} \eta_y \phi w_t y = \tau_t \sum_{y \in \mathcal{Y}} \eta_y \left(\eta_{u,t}(z)\phi w_t y + \eta_{e,t}(z)w_t y \right),$$

where we use the fact that the individual productivity level is independent of the employment status. We deduce that the social contribution is:

$$\tau_t(z) = \frac{\phi \eta_{u,t}(z)}{\eta_{e,t}(z) + \phi \eta_{u,t}(z)} = \frac{1}{1 + \frac{1 - \eta_{u,t}(z)}{\eta_{u,t}(z)\phi}}. \tag{7}$$

The contribution obviously raises with the replacement rate –which is constant– and the population of unemployed agents – that varies along the business cycle.

Fiscal policy. The labor tax τ_t^L finances a quantity G_t of public goods. The government is prevented from raising public debt, such that the government budget is balanced at any date. Formally, the government budget constraint can be expressed as:

$$G_t = \tau_t^L w_t L_t. \tag{8}$$

As a summary an employed agent having a productivity y_t will have a real labor income $\left(1 - \tau_t - \tau_t^L\right) w_t$.

2.5 Agents' program and resource constraints

2.5.1 Sequential formulation

We consider an agent i. She can save in a riskless asset that pays off the post-tax gross interest rate $1 + r_t$. She is prevented from holding too negative savings and the latter must remain greater than an exogenous threshold denoted $-\bar{a}$. At date 0, the agent chooses her consumption

$(c_t^i)_{t\geq 0}$ and her saving plans $(a_t^i)_{t\geq 0}$ that maximize her intertemporal utility, subject to a budget constraint and the previous borrowing limit. Formally, her program can be expressed as follows:

$$\max_{\{c_t^i, a_t^i\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t^i, G_t) \tag{9}$$

$$c_t^i + a_t^i = (1+r_t)a_{t-1}^i + (1 - \tau_t - \tau_t^L)\left(1_{e_t^i=e} + \phi 1_{e_t^i=u}\right)w_t, \tag{10}$$

$$a_t^i \geq -\bar{a}, \tag{11}$$

where $1_{e_t^i=e}$ is an indicator function equal to 1 is $e_t^i = e$ and to 0 otherwise. The budget constraint (10) is very standard and the expression $(1 - \tau_t - \tau_t^L)w_t$ is a compact formulation for the gross (i.e., before-tax) wage of the agent, depending on whether she is employed ($e_t^i = e$) or unemployed ($e_t^i = u$). We now turn to the economy-wide constraints. First, the financial market clearing implies the following relationship:

$$\int_i a_t^i \ell(di) = K_t. \tag{12}$$

The clearing of goods market implies that the total consumption, made of private individual consumption, private firm consumption and public consumption equals total supply, made of output and past capital:

$$\int_i c_t^i \ell(di) + G_t + K_t = Y_t + K_{t-1}. \tag{13}$$

Since every employed agent inelastically supplies one unit of labor, while unemployed agents do not work, the labor L_t in efficient units is defined as:

$$L_t = \eta_{e,t} \sum_{y \in \mathcal{Y}} \eta_y y,$$

since agents have different individual productivities.

Using the transition matrix M_t in equation (2), we deduce the law of motion for the share of unemployed agents $\eta_{u,t}$ is:

$$\eta_{t,u} = 1 - \eta_{e,t} = l_t(1 - \eta_{t-1,u}) + (1 - f_t)\eta_{t-1,u}. \tag{14}$$

The share of agents η_y with productivity y is defined as follows:

$$\eta_y = \sum_{y \in \mathcal{Y}} \eta_{y'} \pi_{y'y}.$$

We can finally formulate our equilibrium definition.

Definition 1 (Sequential equilibrium) *A sequential competitive equilibrium is a collection of individual allocations $(c_t^i, a_t^i)_{t \geq 0, i \in I}$, of aggregate quantities $(G_t, L_t, K_t)_{t \geq 0}$, of price processes $(w_t, w_t, r_t)_{t \geq 0}$, and of social contributions and capital taxes $(\tau_t, \tau_t^L)_{t \geq 0}$, such that, for an initial wealth distribution $(a_{-1}^i)_{i \in I}$, and for initial values of capital stock $K_{-1} = \int_{i \in I} a_{-1}^i \ell(di)$, of capital tax τ_0, and of the initial aggregate shock s_{-1}, we have:*

1. *given prices, individual strategies $(c_t^i, a_t^i)_{t \geq 0, i \in I}$ solve the agents' optimization program in equations (9)–(11);*

2. *financial and good markets clear at all dates: for any $t \geq 0$, equations (12) and (13) hold;*

3. *the government budget constraint (8) and the UI scheme balance (7) hold at any date;*

4. *factor prices $(w_t, r_t)_{t \geq 0}$ are consistent with (5), and (6).*

3   Solving the model with a history-representation

The previous model is a typical heterogeneous-agent model. As time goes by, heterogeneity is unbounded as agents with different idiosyncratic histories have different wealth and consumption. We now provide our projection theory to obtain a finite-dimensional state-space representation. The basic idea is to group agents according to their idiosyncratic histories at any period.

3.1   Partitions

A partition $\mathcal{P}$ can be seen as a collection of set of histories $h \in \mathcal{P}$ such that idiosyncratic history s^t at any period belongs to exactly one element of $\mathcal{P}$: There is one and only one elements $h \in \mathcal{P}$, such that $e^t \in h$. Some direct implications follow. First, there will be some heterogeneity within each element p as, in general, many agents will belong to the same elements[2] h. Second, as idiosyncratic histories change after the realization of the idiosyncratic risk, agents move from an element $h \in \mathcal{P}$ to an element $h' \in \mathcal{P}$ after at any period. When an agent is in an element h in any period t, the probability that it belongs to any element h' the next period is denoted $\Pi_{t+1,h,h'}$ and it can be time-varying in the general case.

The idea of our theory is consider elements of the partition instead of the individual agents. For the sake of concreteness, we now present two types of partitions, explicit partitions used in

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[2] To simplify the discussion, we say that an agent belongs to p, when it has experienced an idiosyncratic history $e^t \in p$.

various papers, and then implicit partition, using insight from the steady-state distribution of wealth.

Explicit partition. First, a finite history of length $N \geq 1$ is a vector $s^N = (s_{-N+1}, \ldots, s_0) \in \mathcal{S}^N$ of length N representing the realizations of idiosyncratic status over the N consecutive previous periods. A first simple construction of a partition is to consider agents with the same value of s^N as being in the same elements of the partition. In words, agents with the same realization of idiosyncratic risk for the last N periods are considered in the same h of $\mathcal{P}$. The number of elements of $\mathcal{P}$ is $(\#\mathcal{S})^N$, which can be a large number.

Even though it has not been formalized in those terms, such partitions have already been used in the literature. First, Challe, Matheron, Ragot, and Rubio-Ramirez (2017) use a three-state partition, which is $(\{e\}, \{eu\}, \{uu\})$. In each period, any agent can be in one and only one of these three states: employed, unemployed now and in the previous period, or unemployed now and employed before. Furthermore, the current employment status uniquely pins down the productivity status. This three-state partition is shown to be sufficient to capture time-varying precautionary savings. The transition matrix between these three states is easy to derive from labor market transitions. For instance $\Pi_{t,\{e\},\{eu\}} = l_t$, $\Pi_{t,\{eu\},\{e\}} = \Pi_{t,\{uu\},\{e\}} = f_t$, and finally $\Pi_{t,\{eu\},\{uu\}} = 1 - f_t$, where we recall that l_t and f_t are the job-transition probabilities –see equation (2).

Second, LeGrand and Ragot (2017) use a more general truncation space in a model where individual productivity is pinned down by employment status.. For a given parameter N, the partition, denoted $\mathcal{P}^N$, contains all idiosyncratic histories of length N, or more formally all vectors $(s_{-N+1}, \ldots, s_0) \in \mathcal{S}^N$. In this case, the transition matrix between partition elements can be easily derived from the transition matrix M_t.

Although intuitive, this construction can generate a huge number of elements h to follow, as their number $(Card\ \mathcal{S})^N$ grows exponentially. Considering all these elements can be inefficient, as we may follow some histories with very low probabilities to occur. As a consequence, using implicit partition appears more efficient in economies with many idiosyncratic states.

Implicit partitions using the steady-state distribution of wealth An implicit partition can be defined using the steady-state distribution of wealth of the model, when there is no aggregate shocks (i.e.. $Z_t = 1$). In this case, it is known that there is, in equilibrium, an invariant steady-state distribution of beginning-of-period wealth $(a, s) \mapsto \Gamma(a, s)$ for $[-\bar{a}; +\infty[$ and $s \in \mathcal{S}$ such that the number of agents having a current state s and a wealth in the interval $[a_0; a_1]$ is $\int_{a_0}^{a_1} d\Gamma(a, s)$.

One can define implicitly a partition as a collections of sets $(B_k)_{k=1,\ldots,K}$, $B_k \subset \mathbb{R}$, such that

$$\begin{cases} [-\bar{a}, +\infty[= \cup_{k=1,\ldots,K} B_k, \\ B_k \cap B_{k'} = \emptyset \qquad\qquad \text{for all } k \neq k'. \end{cases}$$

Elements of $h \in \mathcal{P}$ can be defined as the elements (s, B_k) for $s \in \mathcal{S}$ and $k = 1..K$. In words, we use a partition in the space of wealth for any agent in state s to implicitly defined set of histories.

The transition probabilities $\Pi_{h,h'}$ across elements p and p' can be found by simulations of the model without aggregate shocks. More efficiently, they can be derived from the steady state policy rules. Knowing the policy rule and the beginning of period wealth within the set B_k and the type s, one can derive the end-of-period wealth and then the share of agents moving to any elements $h = (s, B_k)$ to $h' = (s', B_{k'})$.

Introducing aggregate shocks in the previous construction doesn't change the definition of elements of $\mathcal{P}$ (which are given histories) but only transitions probabilities $\Pi_{t,h,h'}$ which become time-varying if and only if the transition matrix M_t, defined in Section 2.2, is time-varying. Indeed, if there s only TFP risk, and no time-varying aggregate risk, then for any idiosyncratic history s^t at period t, the probability to have any other history s^{t+1} at period $t + 1$ is *not* time-varying. As a consequence, transitions $\Pi_{h,h'}$ are constant.

Comparison with Reiter's algorithm.. At this stage, it may be useful to compare the previous construction with the one of Reiter (2009) to clarify the difference. Reiter doesn't follow histories but the time-varying number of agents within given brackets of wealth. After a TFP shock, even if idiosyncratic risk is not time-varying, the number of agents within any bracket of wealth is time-varying, as saving decisions change after a TFP shock. As a consequence, transition probabilities $\Pi_{t,h,h'}$ would be time-varying in Reiter's algorithm. In words, when idiosyncratic risk is not time-varying, in the Reiter's algorithm, the boundaries in terms of wealth defining brackets are constant, and the number of agents in any bracket is time-varying; In our setup, the number of agents within each set is constant (such as transitions among these sets) but the boundaries of any set would be time-varying, as saving decisions are time-varying. This difference will make it possible to solve for optimal policies.

3.2   Projection of the model

We now use a general partition $\mathcal{P}$, express the dynamics of the projected model.

3.2.1 Aggregating variable

The core idea of the model projection consists in following average values for each element $h \in \mathcal{P}$. First, following this line of reasoning, a share of the agents' population is represented by the same history. This share, that we denote as $S_{h,t}$, depends on the history $h \in \mathcal{P}_t$ and on the date t. Formally, the evolution of this population share is defined as follows:

$$S_{h,t} = \sum_{\tilde{h} \in \mathcal{P}_{t-1}} S_{\tilde{h},t-1} \Pi_{t,\tilde{h},h}.$$

It now remains to find explicit law of motions of average values within each set h. For the sake of generality, we consider a generic individual choice variable that we denote X and that can represent savings or consumption for instance. As any individual variable, this variable will depend at date t on the agent's initial saving a, the history of her individual statuses and of aggregate shocks up to date t. The variable will therefore be denoted as $X_t\left(a, s^t, z^t\right)$ at date t.

The projection on the histories consists in averaging the variable among all agents sharing the same history, as if all agents with history h would be represented by a unique representative agent with history h, endowed with the average variable value. More formally, the variable value for the history $h \in \mathcal{P}_t$, which is denoted as $X_{h,t}(z^t)$ or simply $X_{h,t}$, is defined as:

$$X_{h,t} = \frac{\int_a \sum_{e^t \in h} X_t\left(a, s^t, z^t\right) da}{S_{h,t}}. \tag{15}$$

This is the basic projection operation. Other operations are also useful but for the sake of simplicity, we detail them in Appendix A.

3.3 The projected model

We consider the economic model presented in Section 2. The model is characterized by the following set of equations. Denoting by ν_t the Lagrange multiplier associated to the individual budget constraint, we obtain:

$$u'\left(c_t\left(a, e^t, z^t\right)\right) = \beta \mathbb{E}\left[(1 + r_{t+1}(z_{t+1})) u'(c_{t+1}\left(a, e^{t+1}, z^{t+1}\right))\right] + \nu_t\left(a, e^t, z^t\right),$$
$$\tag{16}$$

$$\nu_t\left(a, e^t, z^t\right) \geq 0,$$
$$\nu_t\left(a, e^t, z^t\right)\left(a_t\left(a, e^t, z^t\right) + \bar{a}\right) = 0.$$

The individual budget constraint is:

$$c\left(a, e^t, z^t\right) + a_t\left(a, e^t, z^t\right) = (1 + r_t(z_t))a_{t-1}\left(a, e^{t-1}, z^{t-1}\right) + (1 - \tau_t - \tau_t^L)(1_{e_t^i=e} + \phi 1_{e_t^i=u})w_t(z_t),$$

(17)

while the social contribution τ_t is unchanged and defined in equation (7). Costs of labor w and of labor r have the same expressions as above and are defined in equations (5) and (6) respectively. Finally, the financial market clearing is given by:

$$\int_a \sum_{s^t \in \mathcal{S}^t} a_t\left(a, s^t, z^t\right) da = K_t\left(z^t\right).$$

(18)

We will project the model on a given history partition $\mathcal{P}$. We start with defining the quantity $\tilde{a}_{h,t}$ as follows:

$$\tilde{a}_{h,t} = \frac{1}{S_{h,t}} \sum_{\tilde{h} \in \mathcal{P}} \Pi_{t,\tilde{h},h} S_{\tilde{h},t-1} a_{\tilde{h},t-1},$$

(19)

where $a_{h,t}$ is the average end-of-period asset holding for asset for history h defined as in equation (15). The quantity $\tilde{a}_{h,t}$ is the beginning-of-period wealth of agents with history h that is computed as the average wealth of agents transiting from history $\tilde{h}$ at date $t - 1$ to history h at t. This is similar to a pooling operation, where the wealth of agents, with history $\tilde{h}$ at date $t - 1$ but endowed with history h at t, is pooled together and equally shared among all agents with history h. Using this notation, the individual budget constraint can be expressed as:

$$c_{h,t} + a_{h,t} = (1 + r_t)\tilde{a}_{h,t} + (1 - \tau_t)(\eta_{e,t} + \phi\eta_{u,t})w_t.$$

(20)

This aggregated budget constraint can be interpreted as the budget constraint of a representative agent with history h. The financial clearing equation (18) becomes:

$$K_t = \sum_{h \in \mathcal{P}} S_{t,h} a_{t,h} = \sum_{h \in \mathcal{P}} S_{t+1,h} \tilde{a}_{t+1,h},$$

(21)

where everything happens as if we had $Card\ \mathcal{P}$ agents, each having a weight $S_{t,h}$ and being endowed with wealth $a_{t,h}$. As the previous equations are linear, their projection is simple. This is not the case for the Euler equations. Using the techniques presented in Appendix, We show that one can construct aggregate Euler equations among elements of the partition $\mathcal{P}$, introducing additional variables, which captures the time-varying heterogeneity with each

elements $h \in \mathcal{P}$.The Euler equations (16) become:

$$u'\left(c_{h,t}\right) = \beta\mathbb{E}(1+r_{t+1})\sum_{h'\in\mathcal{P}}\Pi_{t+1,h,h'}\varphi_{t+1,h,h'}u'(c_{h',t+1}) + \frac{\nu_{h,t}}{\xi_{h,t}^{u'}}, \tag{22}$$

$$\text{with: } \varphi_{t+1,h,h'} = \psi_{t,h,h'}^c\frac{S_{h',t+1}}{S_{h,t}}\frac{\xi_{h',t+1}^{u'}}{\xi_{h,t}^{u'}}. \tag{23}$$

The quantity ξ defined in equation (57) reflects the non-linearity of u' in the aggregation , while the quantity ψ defined in equation (59) reflects the aggregation of an expectation function. These additional variables will actually be simple to compute, as we show below.

3.4 Simulating the projected model

3.4.1 The steady-state

We consider a steady-state partition $\mathcal{P}$. We make the assumption that all credit-constrained agents are endowed by a unique particular history denoted h^{cc}.

Assumption A (Credit constrained histories) *There exists an element $h^{cc} \in \mathcal{P}$, such that all agents having histories in h^{cc} are credit constrained, and only them.*

We start with the probability distribution among agents with history $h \in \mathcal{P}$ and asset holding a, denoted $\Gamma^{\mathcal{P}}\left(a,h\right)$ defined over $A\times\mathcal{P}$, where $A = [-\bar{a},\infty)$ is the saving space. This probability captures the heterogeneity in wealth among agents in the same set h. It can be derived from the steady-state distribution Γ of the Bewley model – that is known to exist and to characterize the equilibrium (see Huggett 1993). We can deduce the size of the population of agents with history h, that we denote S_h:

$$S_h = \int_A \Gamma^{\mathcal{P}}\left(da,h\right), \tag{24}$$

which is simply the measure of agents with history h, independently of their asset holdings.

We now turn to the policy functions and asset choices. The beginning-of-period asset holding, denoted $\tilde{a}_h$ is:

$$\tilde{a}_h = \int_A a\frac{\Gamma^{\mathcal{P}}\left(da,h\right)}{S_h},$$

which is the average asset holding among the population with history h. The asset choice, a'_h,

which is the average end-of-period asset holding of agents with history h can be expressed as:

$$a'_h = \int_A g_a(a, e_0(h)) \frac{\Gamma^{\mathcal{P}}(da, h)}{S_h}, \tag{25}$$

where $e_0 : h \in \mathcal{P} \rightarrow e_0(h) \in \{0, 1\}$ returns the current idiosyncratic state for any history h. We proceed similarly for the average consumption choice –denoted c_h– and the average Lagrange multiplier of the credit constraint –denoted ν_h. We obtain the following expressions:

$$c_h = \int_A g_c(a, e_0(h)) \frac{d\Gamma^{\mathcal{P}}(da, h)}{S_h}, \tag{26}$$

$$\nu_h = \int_A \nu(a, e_0(h)) \frac{d\Gamma^{\mathcal{P}}(da, h)}{S_h}. \tag{27}$$

Note that ν_h is positive if and only if a positive measure of agents having history h face binding credit constraints.

We deduce that the steady-state equilibrium is characterized by the following equations:

$$c_h + a_h = (1 + r)\tilde{a}_h + (1 - \tau_t - \tau_t^L)(\eta_{h,e} + \phi\eta_{h,u})w, \tag{28}$$

$$u'(c_h) = \beta(1 + r) \sum_{h' \in \mathcal{P}} \Pi_{h,h'}\varphi_{h,h'}u'(c_{h'}), \text{ for } h \neq h^{cc}, \tag{29}$$

$$a_{h^{cc}} = -\bar{a}, \tag{30}$$

$$\sum_{h \in \mathcal{P}} S_h a_h = K, \tag{31}$$

$$\tilde{a}_h = \sum_{\tilde{h} \in \mathcal{P}} \Pi_{\tilde{h},h} S_{\tilde{h}} a_{\tilde{h}}. \tag{32}$$

The other quantities (w, r, η_u, η_e, and τ) can easily be deduced from their definitions.

The key equation is the Euler equation (29), where the expression of $\varphi_{h,h'}$ is $\varphi_{h,h'} = \psi^c_{h,h'} \frac{S_{h'}}{S_h} \frac{\xi^{u'}_{h'}}{\xi^{u'}_h}$ – see equation (23). As explained above, the terms in $\varphi_{h,h'}$ reflects two elements: (i) the non-linearity of u' in ξ, and (ii) the issues related to conditional expectation aggregation in ψ. This second effect comes the fact that in general, starting from a given history, not all histories can be attained. This effect disappears at the steady-state and the expression of the Euler equation can be simplified, as stated in the following Proposition.

Proposition 1 (Allocation) *The Euler equation can be written as, for $h \in \mathcal{P}$,*

$$\xi_h u'(c_h) = \beta(1 + r) \sum_{h' \in \mathcal{P}^\infty} \Pi_{h,h'} \xi_{h'} u'(c_{h'}) + \nu_h / \xi^{u'}_{h^{cc}}.$$

Moreover, if $\varphi_{h,h'} \to 1$, then $\xi_h \to S_h \xi_h^{u'}$.

Proof. We define $\boldsymbol{\Pi} = (\Pi_{h,h'})_{h,h' \in \mathcal{P}}$, $\boldsymbol{\Pi}^\varphi = (\Pi_{h,h'} \varphi_{h,h'})_{h,h' \in \mathcal{P}}$, $\boldsymbol{\nu} = (\nu_h)_{h \in \mathcal{P}}$, and $\boldsymbol{I}$ the identity matrix of dimension equal to the cardinal of $\mathcal{P}$. The Euler equations (29) for all $h \in \mathcal{P}^\infty$ become under matrix form:

$$\left(\boldsymbol{I} - \beta(1+r)\boldsymbol{\Pi}^\varphi \right) \boldsymbol{u} = \boldsymbol{\nu} / \xi_{h^{cc}}^{u'}.$$

We define the vector $\tilde{\boldsymbol{u}}$ as: $\tilde{\boldsymbol{u}} = \left(\boldsymbol{I} - \beta(1+r)\boldsymbol{\Pi} \right)^{-1} \left(\boldsymbol{I} - \beta(1+r)\boldsymbol{\Pi}^\varphi \right) \boldsymbol{u}$. We can easily check that:

$$\tilde{\boldsymbol{u}} = \beta(1+r)\boldsymbol{\Pi}\tilde{\boldsymbol{u}} + \boldsymbol{\nu} / \xi_{h^{cc}}^{u'}$$

Defining $\xi_h = \tilde{\boldsymbol{u}}_h / u'(c_h)$, we then deduce that for all $h \in \mathcal{P}$, the following equation holds:

$$\xi_h u'(c_h) \quad = \quad \beta(1+r) \sum_{h \in \mathcal{P}^\infty} \Pi_{h,h'} \xi_{h'} u'(c_{h'}) + \nu_h / \xi_{h^{cc}}^{u'}, \tag{33}$$

which proves the first part of the proposition.

For the second part, let us now assume that $\psi_{t,h,h'}^c = 1$. Then $\varphi_{h,h'} = \dfrac{\psi_{h,h'}^c S_{h'}}{S_h \xi_h^{u'}} \xi_{h'}^{u'} = \dfrac{S_{h'} \xi_{h'}^{u'}}{S_h \xi_h^{u'}}$ and:

$$S_h \xi_h^{u'} u'(c_h) = \beta(1+r) \sum_{h' \in \mathcal{P}^\infty} \Pi_{h,h'} S_{h'} \xi_{h'}^{u'} u'(c_{h'}) + \frac{\nu_h}{\xi_{h^{cc}}^{u'}}.$$

We deduce as a consequence (from uniqueness and continuity) that $\xi_h = S_h \xi_h^{u'}$. ∎

3.4.2 Final formulation of the projected model

The previous construction provides a simple approximation procedure to simulate the model with aggregate shocks. The key assumption that we make to make the model easy to simulate is to assume that the steady-state coefficient $(\xi_h)_{h \in \mathcal{P}}$ that are deduced from Proposition 1 remain unchanged in the presence of aggregate shocks. We further assume that agents with history h^{cc} remain the sole credit-constrained agents in the model, even in the presence of aggregate shocks.

We finally obtain the following final formulation of the projected model

$$c_{h,t} + a_{h,t} = (1+r)\tilde{a}_{h,t} + (1 - \tau_t - \tau_t^L)(\eta_{h,e,t} + \phi\eta_{h,u,t})w,$$

$$\xi_h u'(c_{t,h}) = \beta\mathbb{E}(1 + r_{t+1})\sum_{h' \in \mathcal{P}}\Pi_{t+1,h,h'}\xi_{h'}u'(c_{t+1,h'}), \text{ for } h \neq h^{cc}, \qquad (34)$$

$$a_{h^{cc}} = -\bar{a},$$

$$\sum_{h \in \mathcal{P}}S_{t,h}a_{t,h} = K_t,$$

$$\tilde{a}_{t,h} = \sum_{\tilde{h} \in \mathcal{P}}\Pi_{t,\tilde{h},h}S_{t-1,\tilde{h}}a_{t-1,\tilde{h}},$$

which is similar to the initial formulation, except for the Euler equation (34), where the coefficient $(\xi_h)_{h\in\mathcal{P}}$ are deduced from the steady-state (see Proposition 1).

4 Ramsey program

4.1 Optimal policies

We now derive optimal Ramsey policies in the Bewley model. Comparing the Ramsey allocations in our setup with those of a complete insurance-market economy will enable us to identify the specific role of redistribution and the lack of insurance. However, solving for Ramsey policies in the general case is difficult. Indeed, one has to introduce additional state variables, such as the distribution of Lagrange multipliers for the relevant individual constraint.[3] Solving for this joint distribution is particularly difficult.

The main idea of the current method is to solve for the Ramsey optimal policy for the complete model and *then* to project the resulting equations onto a partition $\mathcal{P}$. We first explain the methodology to solve the model and project solutions onto $\mathcal{P}$, we then describe our algorithm for computing Ramsey policies and we finally discuss the relationships with other methods.

The Ramsey problem consists in determining the fiscal policy –here equivalently, public spending G_t and labor tax rate τ_t^L – that corresponds to the "best" competitive equilibrium, according to an aggregate welfare criterion. In other words, the planner has to select fiscal policy and individual choices, subject to government and individual budget constraints (36) and (37), and subject to Euler equations (38) –that guarantee the optimality of individual choices.

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[3]The relevant individual constraint depends on the way the Ramsey problem is written. As we discuss below, in the Lagrangian approach of Marcet and Marimon (2011), these relevant constraints are the individual Euler equations. Bhandari, Evans, Golosov, and Sargent (2016) use a primal approach and thus consider the individual Lagrange multiplier on the budget constraint.

Formally, the Ramsey problem can be written as follows:

$$\max_{\left(\tau_t^L, G_t, (a_t^i, c_t^i)\right)_{t \geq 0}} \mathbb{E}_0 \left[\int_i \sum_{t=0}^{\infty} \beta^t U(c_t^i, G_t) \ell(di) \right], \tag{35}$$

$$G_t \leq \tau_t^L w_t L_t. \tag{36}$$

for all $h \in \mathcal{P}$:

$$c_t^i + a_t^i \leq (1 + r_t) a_{t-1}^i + (1 - \tau_t - \tau_t^L) \left(1_{e_t^i = e} + \phi 1_{e_t^i = u} \right) w_t, \tag{37}$$

$$u' \left(c_t^i \right) = \beta \mathbb{E} \left[(1 + r_{t+1}) u'(c_{t+1}^i) \right] + \nu_t, \tag{38}$$

$$\nu_t (a_t^i + \overline{a}) = 0, \tag{39}$$

$$K_t = \int_i a_t^i \ell(di), \ L_t = \eta_{e,t} \sum_{y \in \mathcal{Y}} \eta_y y, \tag{40}$$

$$c_{h,t}^i, (a_{h,t}^i + \overline{a}) \geq 0. \tag{41}$$

Other constraints are the evolution equation of the population shares of employed and unemployed agents (14) and the definition of the social contribution τ_t (7).

It is easy to derive first-order conditions. We discuss in Section 4.2 below issues related to second-order conditions. First, we denote by $\beta^t \lambda_t^i$ the discounted Lagrangian multiplier of the Euler condition (37) for agent i, and by $\beta^t \mu_t$ the Lagrangian multiplier on the government budget constraint (36). The Lagrange multiplier λ_t^i measures how costly it is for the planner to internalize the Euler equation. To ease the interpretation of first-order conditions, we introduce the following notation:

$$\psi_t^i = u' \left(c_t^i \right) - \left(\lambda_t^i - \lambda_{t-1}^i (1 + r) \right) u'' \left(c_t^i \right), \tag{42}$$

which is the *social valuation of liquidity of agent i*. Indeed, if agent i receives one additional unit of goods today, this additional unit will be valued $u' \left(c_t^i \right)$. This value only accounts for private valuation, but should also include the effect on the internalization cost of Euler equations. Indeed, this additional unit affects agent's incentive to save from period $t - 1$ to period t and from period t to period $t + 1$. This effect is captured by the second term at the right hand side, proportional to $u''(c_t^i)$.

We now provide the expressions of the first order conditions of the Ramsey program. Of note, as we discuss in Section 4.2, these conditions are necessary, but not sufficient, to guarantee

the existence of an internal solution.

$$\mu_t = v'(G_t) \tag{43}$$

$$\mu_t L_t = \sum_{s^t \in \mathcal{S}^t} \int_a (1 - \tau_t - \tau_t^L) \left(\phi 1_{e_0(s^t)=u} + 1_{e_0(s^t)=e} \right) \psi_t \left(a, s^t, z^t \right) da, \tag{44}$$

$$\psi_t \left(a, s^t, z^t \right) = \beta \mathbb{E}_t \left[(1 + r_{t+1}(z^{t+1})) \psi_{t+1} \left(a, s^{t+1}, z^{t+1} \right) \right] \tag{45}$$

where $e_0(s^t)$ denotes the current employment status of agent with individual status s^t.

4.2 Remark on the convexity of the program

A traditional problem with Ramsey program is that the set of feasible allocations is not convex. This problem is quite general and also exists in a representative-agent economy. The non-convexity is related to constraint associated to Euler equation – which is neither convex nor linear. Therefore, if first-order conditions are still necessary, they may be non-sufficient and generate three different types of problems: 1) the first order condition may characterize a local minimum; 2) the steady-state solution may not exist; 3) multiple equilibria may exist.

The first concern can be easily addressed, for instance by checking that small variations around the solution allocation do not yield a higher aggregate welfare. The second concern has been raised by Straub and Werning (2014), who show that in some cases the solution of the planner may not be an interior solution with constant real variables.[4] The possibility to solve the model with perturbation methods helps solve this issue. Indeed, studying the behavior of the model after perturbing the steady state with small aggregate shocks, provides insight regarding the subsequence convergence –or not– toward the interior solution. The last concern is more difficult to properly address. Up to our knowledge, the only imperfect solution consist sin exploring the convergence for various initial values and checking that the local maximum is indeed a global one.

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[4]Recent contributions such as Chari, Nicolini, and Teles (2016) show that the behavior of Lagrange multipliers depends on the set of instruments available to the planner. In addition, Chen, Chien, and Yang (2017) show theoretically that in an incomplete insurance-market model that the solution is interior.

4.3 Projecting Ramsey conditions

We follow the same path as in Section 3 for the Ramsey program. We start with two definitions:

$$\lambda_{h,t} = \int_a \sum_{s^t \in h} \lambda_t \left(a, s^t, z^t \right), \tag{46}$$

$$\Lambda_{h,t} = \frac{\sum_{h' \in \mathcal{P}_{t-1}} S_{h',t-1}\lambda_{h',t-1}\Pi_{h',h,t}}{S_{h,t}}, \tag{47}$$

$$\psi_{h,t} = u' \left(c_{h,t} \right) - \left(\lambda_{h,t} - \Lambda_{h,t}(1 + r_t) \right) u'' \left(c_{h,t} \right) + \kappa_{h,t}^1. \tag{48}$$

The quantities $\lambda_{h,t}$ and $\psi_{h,t}$ have straightforward interpretations. They are the history counterparts of the Lagrange multiplier λ and of the liquidity valuation ψ. The variable $\Lambda_{h,t}$ in equation (47) new and is the average per capita cost of internalizing the previous period Euler equation for agents with history h today. Roughly speaking, this is the past average of past values of Lagrange multiplier $(\lambda_{h,t-1})_{h \in \mathcal{P}}$. The residual $\kappa_{h,t}^1$ is the time-varying error in the aggregation procedure. This term captures both time varying heterogeneity within each element of the partition and the time-varying correlation between Lagrange multipliers and marginal utilities. The exact expression of this residual isn't relevant, as we will make simplifying assumptions to simulate the model.

Aggregating the Ramsey first-order equations (43)–(45) yields:

$$\mu_t = v' \left(G_t \right), \tag{49}$$

$$\mu_t L_t = (1 - \tau_t - \tau_t^L) \sum_{h \in \mathcal{P}} S_{h,t} \left(\phi \eta_{u,t} + \eta_{e,t} \right) \psi_{h,t} + \kappa_t^2, \tag{50}$$

$$\psi_{h,t} = \beta \mathbb{E}_t (1 + r_{t+1}) \psi_{t+1,h,h'}^{\psi} \psi_{h,t+1}. \tag{51}$$

The residual κ_t^2 captures the time-varying heterogeneity within elements of the partitions, concerning $\psi_{h,t}$ Additionally, we also have an aggregated version of the budget constraint (20), of the Euler equation (22), of the financial market clearing (21). These equations are the same as in the initial aggregation of Section 3.2. As in the initial aggregation, a pooling-like equation (19) also holds.

However, as in Section 3.4, further assumptions are needed for allowing us to simulate the model. The previous equations are exact first-order conditions for the optimal Ramsey policy. The approximation to simulate the model is the assumption that t $\kappa_t^1 = \kappa_t^2 = 0$, $\psi_{t+1,h,h'}^{\psi} = 1$, and $\varphi_{t+1,h,h'}$ remains constant. In words, we neglect time-varying heterogeneity within elements of $\mathcal{P}$. As in the initial projection, we deduce the quantities $(\varphi_{h,h'})_{h,h'}$ from the steady-state.

The equations that we simulate are the following ones: and then simulate

$$\Lambda_{h,t} \equiv \frac{\sum_{h' \in \mathcal{P}} S_{h',t-1} \lambda_{h',t-1} \Pi_{h',h,t}}{S_{h,t}},$$

$$\psi_{h,t} = u'(c_{h,t}) - (\lambda_{h,t} - \Lambda_{h,t}(1+r_t)) u''(c_{h,t})$$

and

$$\Lambda_{h,t} = \frac{\sum_{h' \in \mathcal{P}} S_{h',t-1} \lambda_{h',t-1} \Pi_{h',h,t}}{S_{h,t}}, \tag{52}$$

$$\psi_{h,t} = u'(c_{h,t}) - (\lambda_{h,t} - \Lambda_{h,t}(1+r_t)) u''(c_{h,t}) \tag{53}$$

$$\mu_t = v'(G_t), \tag{54}$$

$$\mu_t L_t = (1 - \tau_t - \tau_t^L) \sum_{h \in \mathcal{P}} S_{h,t} (\phi \eta_{h,e} + \eta_{h,u}) \psi_{h,t}, \tag{55}$$

$$\psi_{h,t} = \beta \mathbb{E}_t (1 + r_{t+1}) \psi_{h,t+1}, \tag{56}$$

without mentioning history budget constraints, history Euler equations, pooling-like equations, financial market clearing condition, as well as the definitions of factor prices r and w, of the social contribution τ_t, and the dynamics of population shares $\eta_{e,t}$ and $\eta_{u,t}$.

4.4 Comparison with other methods

To our knowledge, only three other papers provide general solution method to derive optimal Ramsey policies in incomplete insurance-market models.

First, Açikgöz (2015) provides an algorithm to solve for the steady-state allocation of the Ramsey program. He assumes some specific functional forms and show the convergence of the algorithm. This is a way to find the joint distribution over Lagrange multipliers and initial asset holdings. At this stage, we are not aware of any application of this algorithm to an economy with aggregate shocks.

Second, Nuño and Moll (2017) use a continuous-time approach and mean-field games to characterize optimal steady-state allocations. Their algorithm develops a projection method to characterize the relevant value functions and Lagrange multipliers. Our solution makes a more extensive use of the steady-state properties of the Bewley model, that enables us to properly distort the projection on a relevant grid. Although our model is expressed in discrete time, a methodology similar to ours can be applied to continuous-time models. An additional gain of

our method in discrete time is that introducing aggregate shocks is straightforward, as we have seen above.

Third, Bhandari, Evans, Golosov, and Sargent (2016) present a solution method for models with aggregate shocks. Their solution relies on perturbation methods around time-varying allocations (and not around the steady-state). They solve the model by approximating the actual distribution by a very large number of agents. As we use more extensively the steady-state properties of the Bewley model, we can simulate the economies with a very small number of agents –see Section 5. As a consequence, our solution allows us to study Ramsey problems with a number of instruments.

5   Numerical examples

We now provide two numerical examples to apply the previous setup. In the first one, we consider zero public spendings (as in the model projection of Section 3) and thus we do not solve for optimal policies. We compute the dynamics of the model with TFP shocks and compare the current algorithm with alternative ones, such as Krusell and Smith (1998). This analysis is done to show how our methodology compares to other ones in a standard model with aggregate shocks. In the second example, we solve for the optimal fiscal policy to investigate the optimal provision of the public good G_t along the business cycle. These examples are chosen to be simple, and to enable us as to present in a very transparent framework the properties of the methodology we propose.

5.1   Calibration of the model

We provide a quarterly calibration of the model that follows the parameter choice of Krueger, Mitman, and Perri (2017). This calibration aims at reaching three two of moments. First, the process for the aggregate shocks is set to match the probability of severe recessions in the US postwar economy, 1948-2014.III. A severe recession is defined as a period where the unemployment rate is above 9. The frequency of severe recessions is 16.48% with an expected length of 22 quarters. The average unemployment rate during these recessions is 8.39%, whereas the average unemployment rate in good times is 5.33%. The average drop in GDP per capita is 7% compared to normal times.

In the benchmark model, this dynamics is captured by the following AR(1) process for TFP

shocks:[5]

$$z_t = (1 - \rho_z) + \rho_z z_{t-1} + \varepsilon_t^z,$$

where $\varepsilon_t^z \sim \mathcal{N}(0, \sigma_z^2)$, with $\rho_z = 0.9464$ and $\sigma_z = 0.4\%$.

Unemployment risk. As explained above, we assume that both the job separation and job finding rates are functions of current and past values of the TFP level. It should be clear that this assumption is made here to compare our simulation techniques with standard methods.[6]

We have the following processes for employment transition probabilities:

$$\begin{aligned} f_t &= f_{SS} + \rho_0^f z_t + \rho_1^f z_{t-1}, \\ l_t &= l_{SS} + \rho_0^l z_t + \rho_1^l z_{t-1}, \end{aligned}$$

where f_{SS} and l_{SS} are set to their postwar values. We find $f_{SS} = 0.786$ and $l_{SS} = 0.048$. The postwar dynamics is consistent with $\left(\rho_0^f, \rho_1^f\right) = (3.483, 0.436)$ and $\left(\rho_0^l, \rho_1^l\right) = (-0.613, 0.219)$. As expected and as is consistent with the labor literature, the job separation rate is less volatile than the job finding rate and is countercyclical on impact. Finally, following standard estimates, the replacement rate is assume to be $\phi = 50\%$.

Productivity risk. The income risk conditional on employment is estimated by Krueger, Mitman, and Perri (2017) using annual data. Translating these esimates into quarterly values (with the same autocorrelation and variance), we deduce the following process:

$$\log y_t = \rho^y \log y_{t-1} + \varepsilon_t^y,$$

with $\varepsilon_t \sim \mathcal{N}(0, \sigma_z^2)$. We find an autocorrelation equal to $\rho^y = 0.9923$ and a variance equal to $\sigma_y^2 = 0.0098$. The Rouwenhorst procedure is then used to discretize the process $\log y_t$ into a seven-state Markov process. As agents can be either employed and unemployed, each agent can be in $14 = 7 \times 2$ idiosyncratic states.

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[5]Krueger, Mitman, and Perri (2017) estimate this process with a two-state Markov process, in order to use the Krusell and Smith (1998) algorithm. Our AR(1) representation is consistent with their estimation. In Appendix, we provide their estimated values for the sake of comparison.

[6]We could easily simulate the model with many aggregate shocks, as in Challe, Matheron, Ragot, and Rubio-Ramirez (2017) to get more realistic second-order moments compared to the data. We do not follow this route here to focus on a simple case.

Parameter	Description	Value
β	Discount Factor	0.99
α	Capital share	0.36
δ	Depreciation rate	0.025
ϕ	Replacement rate	0.5
f_{SS}	Average job finding rate	0.786
l_{SS}	Average job separation rate	0.048
ϕ	Replacement rate	0.5
ρ^z	Autocorrelation TFP	0.9464
σ_z	Standard deviation TFP shock	0.004
$\left(\rho_0^f, \rho_1^f\right)$	Corr. job find. rate with TFP and TFP(-1)	$(3.483, 0.436)$
$\left(\rho_0^l, \rho_1^l\right)$	Corr. job sep. rate with TFP and TFP(-1)	$(-0.613, 0.219)$
ρ^y	Autocorrelation idio. productivity	0.9923
σ^y	Standard dev. idio productivity	0.0990

Table 1: Parameter values. See main text for descriptions and targets

Production function. The production function is $F(K, L) = ZK^{\alpha}L^{1-\alpha} - \delta K$, where the capital share is $\alpha = 0.36$ and the depreciation rate is $\delta = 0.025$.

Preferences. We assume a log period utility function ($\sigma = 1$) and a discount factor equal to $\beta = 0.99$ to obtain a realistic capital-output ratio.

Table 1 provides a summary of the model parameters. These parameters are standard for quarterly parametrization. Most of the parameters are taken from Den Haan (2010). They are used by Winberry (2016) to compare model outcomes with different computational methods. We consider here the case where the preference for the public good is 0, such that $G_t = \tau_t^L = 0$. As a consequence, we solve for the standard model with TFP shock and without taxes on capital.

5.2   Model outcome without public goods

We first study the model outcome without any provision of public good. In this case, $G = \tau^L = 0$. The model is then a standard Krusell and Smith (1998) type of model, where we do not solve

for optimal policies. This model can be solved with standard techniques, which allows us to compare our solution techniques to standard ones.

We start with computing the Bewley model, which corresponds to the previous model in absence of aggregate shocks. This allows us to determine the steady-state equilibrium. In this case, $\sigma^z = 0$, $f_t = f_{SS}$ and $l_t = l_{SS}$. Using steady-state equilibrium results, we compute the values for the ξ_i parameters that enable to exactly reproduce the distributions of wealth and of consumption on the projected partition.

We use an implicit partition in the space of history using the steady-state wealth distribution. We consider ten brackets of wealth. All constrained agents, independently of their type, belong to the first bracket of wealth, while and the remaining 9 brackets of wealth have roughly the same size.[7] We then consider the distribution of any of the 14 different types of agents among these brackets. We remind that each agent can be in one of 14 different individual states. As a consequence, the partition has $140 = 7 \times 2 \times 10$ different elements. We normalize the average value of ξ_h to 1. We find that the standard deviation across the different sets of histories is 1.62%, which is small.

First, we report the steady state distribution of wealth in the model and in the data, using the the Survey of Consumer Finance 2007, so as to avoid considering the temporary effects of the 2008 financial crisis. The distribution labeled "Model" is both the distribution of the Bewley model and the distribution of the projected model (by construction). The Gini coefficient of wealth distribution generated by the model is high, and equals 0.72 and close to its empirical counterpart value of 0.78. The next Table reports the distribution of wealth for the various quintiles and for the top decile of the wealth distribution. The distribution of wealth is close to its empirical counterpart. The model fails to reproduce the concentration of wealth at the top of the distribution, what is a well-known feature of these models in the literature. Heterogeneous discount factors or heterogeneity in the return to human capital (entrepreneurship) helps reproduce this concentration. We checked that the introduction of aggregate shocks does not alter the shape of the average distribution of wealth. The average Gini coefficient in the model with aggregate shocks and in the Bewley model are indeed very close (0.726 and 0.725).

Second, we simulate the model with aggregate shocks to compute second-order moments. For the sake of comparison, we solve the model with three solution techniques. First, we simulate the model using Krusell-Smith (K-S, henceforth) solution technique. More precisely, we use

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[7]Steady state capital stock is 34.70 and the wealth thresholds to define the brackets of wealths are (0.0100;0.2367; 0.3918; 1.7995; 5.2570; 12.3237; 24.0465; 50.0610; 93.6606; 814.5871). The last value is the highest amount of wealth held by any agent in the model.

	Q1	Q2	Q3	Q4	Q5	D10	Gini
Model	0.1	0.9	5.3	18.3	75.5	69.9	0.72
SCF 2007	−0.2	1.2	4.6	11.9	82.5	53.7	0.78

Table 2: Distribution of wealth

Moments	K-S	Model(sim)	Model(theory)	Description
$sd(Y_t)$	8.81	8.87	8.48	Standard deviation of output
$sd(C_t)$	5.43	5.57	5.33	Standard deviation of output
$sd(L_t)$	1.08	1.12	1.08	Standard deviation of output
$sd(w_t)$	3.32	3.09	2.94	Standard deviation of output
$sd(r_t)$	0.06	0.05	0.05	Standard deviation of output
$corr(Y_t, C_t)$	0.92	0.96	0.95	Correlation output and consumption
$corr(Y_t, L_t)$	0.99	0.99	0.99	Correlation output and labor

Table 3: Comparing second-order moments with different resolution techniques

the Krueger, Mitman, and Perri (2017) algorithm, based on the computational strategy of (2010) and (2010). This algorithm uses projection methods to solve for the optimal policies and simulation techniques to iterate on an aggregate law of motion in capital. Second, we solve for the dynamics of the projected model using DYNARE and simulate the model for 3500 periods to obtain second-order moments (a number of periods consistent with the simulation of the K-S model). Third, we solve the model using the DYNARE solver to find theoretical second-order moments. This last possibility is available because of the structure of our model, which has a high but finite number of equations to simulate. The difference between the last two economies allows us to identify sampling errors in our model. Results are reported in Table 3.

The moments generated by the three methods are very similar to each other. The simulated moments are very close between the K-S economy and the simulation of the projected model. The theoretical moments are a little bit different from the simulated moments, due to sampling errors. The possibility to easily derive theoretical moments is an advantage of the simulation technique we use.

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Appendix

A   Different projection mechanisms

In addition to the basic mechanism described in Section 3.2, we also need to further "tools" to proceed with the projection of the full model.

The first one is to aggregate transformation of variables, such as $f(X)$ –where the function f is assumed to be well-defined for any realization of X. If f is affine, the aggregation of the transformation is simply the transformation of the aggregation. But in more general cases, we need to account for this non-linearity. We will denote the aggregation of a transformation as follows:

$$\int_a \sum_{s^t \in h} f\left(X_t\left(a, s^t, z^t\right)\right) da = S_{h,t} \xi_t^f f\left(X_{h,t}\right),$$

$$\text{where: } \xi_t^f = \frac{\int_a \sum_{s^t \in h} f\left(X_t\left(a, s^t, z^t\right)\right) da}{f\left(X_{h,t}\right)}. \tag{57}$$

This definition is obviously a straightforward rewriting of the aggregation equality. The quantity ξ_t^f has the advantage to concentrate the non-linearity effect. Consistently with what we said before, we have $\xi_t^f = 1$ whenever f is affine.

The second mechanism consists in aggregating the past choices of an agent with a given history. More formally, let consider an history $h \in \mathcal{P}_t$. The previous-period value of the variable X are $X_{t-1}\left(a, s^{t-1}, z^{t-1}\right)$, where possible successors of e^{t-1} are represented by history h. The

aggregation of these previous-period values amounts to:

$$\int_a \sum_{s^t \in h} X_{t-1}\left(a, s^{t-1}, z^{t-1}\right) da = \sum_{\tilde{h} \in \mathcal{P}_{t-1}} \Pi_{t,\tilde{h},h} \int_a \sum_{s^{t-1} \in \tilde{h}} X_{t-1}\left(a, s^{t-1}, z^{t-1}\right) da,$$

$$= \sum_{\tilde{h} \in \mathcal{P}_{t-1}} \Pi_{t,\tilde{h},h} S_{\tilde{h},t-1} X_{\tilde{h},t-1}.$$

In other words, the aggregation of past values is straightforward: Aggregating over past individual choices is equivalent to average past aggregate values.

The third mechanism that we need is to aggregate individual expectations. We again consider a history $h \in \mathcal{P}_t$ and are interested in the aggregation of the expectation $\mathbb{E}\left[X_{t+1}\left(a_0, s^{t+1}, z^{t+1}\right) | s^t\right]$ of next-period variable X, where s^{t+1} is a successor of an element of h. Formally, we have:

$$\int_a \sum_{s^t \in h} \mathbb{E}\left[X_{t+1}\left(a, s^{t+1}, z^{t+1}\right) | e^t\right] da = \mathbb{E}\left[\sum_{h' \in \mathcal{P}_{t+1}} \Pi_{t+1,h,h'} \psi^X_{t+1,h,h'} S_{h',t+1} X_{h',t+1}\right], \tag{58}$$

where:
$$\psi^X_{t+1,h,h'} = \frac{\sum_{s^t \in h} \sum_{s^{t+1} \in h', s^{t+1} \succeq s^t} \int_a X_{t+1}\left(a, s^{t+1}, z^{t+1}\right) da}{\Pi_{t+1,h,h'} S_{h',t+1} X_{h',t+1}}. \tag{59}$$

The notation $s^{t+1} \succeq s^t$ means that the $t+1$-history s^{t+1} is a successor of t-history s^t. The term ψ^X reflects the fact that the left-hand side of (58) only involves the histories h' that can be attained from the history h, while the right-hand side involves *all* histories h', no matter whether they can be attained from h or not.

The last mechanism consists in aggregating the product of two variables X and Y. Again, since the product is by essence non-linear, we need to introduce a corrective term. More formally:

$$\frac{1}{S_{h,t}} \int_a \sum_{s^t \in h} X_t\left(a, s^t, z^t\right) Y_t\left(a, s^t, z^t\right) da = X_{h,t} Y_{h,t} + \eta_t^{X,Y},$$

where: $\eta_t^{X,Y}\left(z^t\right) = \frac{1}{S_{h,t}} \int_a \sum_{s^t \in h} \left(X_t\left(a, s^t, z^t\right) - X_{h,t}\right)\left(Y_t\left(a, s^t, z^t\right) - Y_{h,t}\right) da.$

Again, introducing the quantity $\eta_t^{X,Y}$ will prove to be useful in the model aggregation.

B   Derivation of the History-Representation of the First-Order conditions

$$\max_{\left(r_t, w_t, G_t, (a_t^i, c_t^i)\right)_{t\geq 0}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\sum_{h \in \mathcal{P}} S_h \eta_h u\left(c_h\right) + v\left(G_t\right) \right) \right], \tag{60}$$

$$G_t + K_{t-1} r_t + L_t w_t \leq K_{t-1}^\alpha L_t^{1-\alpha} - \delta K_{t-1} \tag{61}$$

for all $h \in \mathcal{P}$:

$$c_{h,t} + a_{h,t} \leq (1 + r_t)\, \tilde{a}_{h,t} + w_t\left(h\right) \tag{62}$$

$$\xi_h u'\left(c_{h,t}^b\right) = \beta(1 + r_t)\mathbb{E}_{h'}\left(\xi_{h'} u'\left(c_{h',t+1}\right)\right) + \nu_h \tag{63}$$

$$\tilde{a}_{h,t+1} = \sum_{h' \succeq g} \Pi_{g,h,t} a'_{g,t} \frac{S_{g,t}}{S_{h,t+1}} \tag{64}$$

$$K_t = \sum_{h \in \mathcal{P}} S_{h,t} a_{h,t} \Lambda_t, \ \ L_t = (1 - s_t) L_{t-1} + f_t\left(1 - L_t\right), \tag{65}$$

$$c_{h,t}^i, (a_{h,t}^i + \bar{a}) \geq 0, \tag{66}$$

The Lagrangian can be written as

$$J = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{h \in \mathcal{P}} S_{h,t} \eta_h u(c_{h,t}) - \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{h \in \mathcal{P}} S_{h,t} \lambda_{h,t} \tag{67}$$

$$\times \left(\xi_h u_c(c_{h,t}) - \nu_{h,t} - \beta \mathbb{E}_t \left[\sum_{h' \in \mathcal{P}} \Pi_{h,h',t+1} \xi_{h'} u_c(c_{h',t+1})(1 + r_{t+1}) \right] \right)$$

Define

$$\Lambda_{h,t} = \frac{\sum_{h' \in \mathcal{P}} S_{h',t-1} \lambda_{h',t-1} \Pi_{h',h,t}}{S_{h,t}}, \tag{68}$$

Hence

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \sum_{h \in \mathcal{P}} S_{h,t} \left(\eta_h u(c_{h,t}) + \xi_h u_c(c_{h,t}) \left(\Lambda_{h,t}(1 + r_t) - \lambda_{h,t} \right) \right) \tag{69}$$

$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \mu_t \beta^t \left(K_{t-1}^\alpha L_t^{1-\alpha} - \delta K_{t-1} - G_t - K_{t-1} r_t - L_t w_t \right),$$

with

$$c_{h,t} + a_{h,t} \ \leq \ (1 + r_t)\, \tilde{a}_{h,t} + w_t\left(h\right)$$

$$\tilde{a}_{h,t+1} \ = \ \sum_{h' \succeq g} \Pi_{g,h,t} a'_{g,t} \frac{S_{g,t}}{S_{h,t+1}}$$

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Derivative with respect to w_t.

We obtain:

$$\mu_t L_t = \sum_{h \in \mathcal{P}} S_{h,t} \frac{w_{h,t}}{w_t} \psi_{t,e^N}. \tag{70}$$

Then the first-order conditions of the Ramsey program can be written as

$$\mu_t = v'(G_t)$$

$$\psi_{t,h} = \beta \mathbb{E}_t \left((1 + r_{t+1}) \sum_{h' \in \mathcal{P}} \Pi_{t,h,h'} \psi_{t+1,h'} \right), \text{for } h \neq h_{cc}$$

$$\mu_t L_t = \sum_{h \in \mathcal{P}} S_{h,t} \frac{w_{h,t}}{w_t} \psi_{t,e^N}.$$

Note that $\frac{w_{h,t}}{w_t} = \phi$ if the agents is unemployed and $\frac{w_{h,t}}{w_t} = 1 - \tau_t$ is the agent is employed.