When Ramsey Searches for Liquidity

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Abstract

What are the optimal policies in an economy with endogenous liquidity frictions? In this paper, liquidity constraints arise because of costly search-and-matching of non-government issued assets. Government bonds, on the other hand, are fully liquid. I show how to characterize the optimal level of government debt, capital tax (or subsidy), and the initial price level, given an initial level of capital stock. There are two main lessons learned. Taxes on capital converges to a level that balances the trade-off between the efficiency of financing government expenditures and consumption inequality (due to search frictions). Most importantly, a long-run optimal debt-to-GDP ratio can be independent of the initial level of capital stock. A calibrated exercise shows that it should be around 66%. The paper suggests that those countries, which have accumulated debt close to 100% of their GDP since the recent financial crisis, should not permanently roll over their debt.

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1 Introduction

For many advanced economies, government debt was relatively low and stable before the 2008 world-wide financial crisis. In 2010, their debt-to-GDP ratios were more than 80%

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(see Table 1), including the United States (99.4%), the United Kingdom (81.81%), Germany (81.02%), France (85.20%), Italy (116.42%), and many others not listed.[1] While many countries continued to issue more government debt into 2016, Germany had a debt-to-GDP ratio that was closer to the pre-crisis level.

Table 1: Government Debt to GDP Ratios in Percents

	The US	The UK	Germany	France	Italy
2007	64.80	43.52	63.60	64.41	99.68
2010	99.41	81.81	81.02	85.20	116.42
2016	105.31	89.14	70.13	98.22	135.51

It is well understood that government debt is used for tax smoothing purposes. However, the experience in the past world-wide crisis suggests that another important function of government debt is liquidity. For example, it can be used as collateral to borrow since it is usually the most liquid assets (besides fiat money). Is there an optimal level of government debt in the long run when a government provides liquidity service? How should a government use tax along with liquidity provision? In the aftermath of financial crisis faced by many advanced economies, should a government roll over its debt after stabilizing the financial market? Or should a government cut back debt (like Germany) to some long-run target?

This paper attempts to answer these questions in an almost standard neoclassical growth model with endogenous liquidity frictions. There are two types of liquidity in this economy. Privately-issued assets are subject to costly search and matching, while nominal government bonds are fully liquid. I analyze the optimal policy under commitment (a Ramsey plan). I show that under some empirically plausible conditions, there exist a unique long-run debt-to-GDP ratio *regardless of the initial condition*.

To my knowledge, this paper contributes to our understanding of how a government can reduce endogenous financial frictions while at the same time finance expenditures in the least costly way. The portfolio of the national savings, in privately-issued assets (backed by capital stock) and in government bonds, is at the center stage of the analysis. One implication is that in the aftermath of economic disturbances (e.g. financial crisis), a government with a large amount of debt should bring its debt-to-GDP ratio to a unique long-run target.

In the model, private agents face idiosyncratic investment risks. Some of them have investment projects, while the others do not. There is a government that has a exogenous stream of expenditures to finance. It can tax income from labor and capital, and it can issue

[1] The data of US debt is obtained from the Z1 flow-of-funds report published by Federal Reserve Board. The US GDP data is obtained from the FRED data source maintained by Federal Reserve at St. Louis. Eurostat contains both debt and GDP data of European Countries.

government debt with one period maturity.

When private agents have investment projects, they seek outside financing. But because of the search frictions, these agents are constrained in issuing new claims and/or resaling old claims. When agents do not have projects, they can accumulate both government bonds and private claims (issued by those who have investment projects).

Government bonds are fully liquid, but privately-issued assets are partially liquid due to search frictions. Therefore, government bonds provide liquidity service; investors of private claims will demand a return that has a liquidity premium over government bonds, in order to compensate their illiquidity. Notice that, because of costly search and matching, those private agents who are liquidity constrained consume less than those who are not constrained. Government bonds are held due to precautionary motives to reduce the degree of liquidity constraints in the future.

The government faces a key trade-off when providing liquidity. The benefit and cost of more government bonds can be explained by the following. A higher level of (real) government debt is helpful because agents can hold more liquid assets to insure against future liquidity risks, a reason for redistribution. For the government, it is a good time to issue debt when precautionary motives drive down real interest rate. Nevertheless, a higher level of debt also implies that a higher taxation and/or a higher real interest rate in the future, reducing efficiency. This means that providing liquidity is subject to the consideration of redistribution and efficiency.

Search frictions literally exist in many markets, such as markets for corporate bonds, IPO, and acquisition. It can also capture many aspects of frictional financial markets with endogenous market participation (see e.g., Vayanos and Wang, 2013; Rocheteau and Weill, 2011), while still keeping the simple structure of neoclassical macro framework. This tractability is crucial since one can use all the insight from a standard Ramsey plan. In particular, I use the "primal approach" (see e.g., Lucas and Stokey, 1983; Chari and Kehoe, 1999) to show the allocations chosen by a Ramsey planner. The "primal approach" is useful, since one can substitute out all prices and taxes and only focus on quantities that should be chosen by the planner. Then, using the quantities, I can back out the instruments including the level of government debt, the capital tax, and the initial price level at time 0.

Compared to previous studies (such as Barro, 1979; Lucas and Stokey, 1983; Chari, Christiano, and Kehoe, 1994; Aiyagari, Marcet, Sargent, and Seppälä, 2002), the long-run level of government debt does not depend on the initial level of debt any more. There exists an optimal level of debt because of the trade-off when the government provides liquidity. The trade-off is also reflected in the determinants of initial price level. When a government is endowed with some debt at time 0, a higher nominal price level washes out the burden on the government. However, it also reduces the liquidity held by private agents who will use it for financing valuable physical investment at time 0. The initial price level is thus an endogenous object, which is linked with the long-run optimal level of debt.

Notice that the long-run optimal level of debt may still depend on the initial level of capital stock endowed by the economy. This is intuitive because of the Ramsey plan committed from time 0. However, I show that under a class of conventional utility functions, the long-run debt-to-GDP ratio is uniquely determined given an expenditure-to-GDP target. Therefore, initial capital stock does not have an impact on the optimal debt-to-GDP ratio. Such result implies that in the aftermath of economic disturbances, a government that accumulates a large amount of debt should gradually cut its debt back to an optimal level.

A calibrated exercise shows that the debt-to-GDP ratio should be around 66%. As agents' elasticity of intertemporal substitution (EIS) falls, they want to save more and become less sensitive to changes in liquidity premium. The Ramsey planner should provide more government debt to satisfy their saving needs and also to take advantage lower interest rate from precautionary motives. Nevertheless, even for EIS to be 1/10, the optimal debt-to-GDP ratio does not exceed 90%, highlighting the cost of liquidity provision.

When the debt-to-GDP ratio is unique, capital tax is also uniquely set to achieve this goal. In fact, I show that capital tax only depends on the degree of risk sharing between agents with investment opportunities and those without investment opportunities. Interestingly, when search frictions disappear, the economy has full risk-sharing and capital tax should be set as zero, a reminiscent of Judd (1985), Chamley (1986), and Zhu (1992).

This paper has financial frictions in the form of liquidity frictions similar to Kiyotaki and Moore (2012) and Shi (2015)[2]. In contrast to exogenous asset saleability frictions, the liquidity frictions in this paper are endogenize through directed search frameworks in Cui and Radde (2016) and Cui (2016) so that the supply of government debt can affect the participation in asset markets. Recent work by Lagos and Rocheteau (2008), Rocheteau (2011), and Cao and Shi (2014) also use search to endogenize liquidity and asset price, but not on the linkage between asset saleability and asset price as in this paper. Endogenous saleability gives rise to different degree of liquidity constraints and risk sharing.

The presence of liquidity constraints opens up the possibilities for government bonds or fiat money to circulate, which at least goes back to Holmström and Tirole (1998). That is, if private liquidity is not enough, public liquidity can be added to achieve efficiency.[3]

[2]Similar papers at least include Nezafat and Slavik (2010), Del Negro, Eggertsson, Ferrero, and Kiyotaki (2017), Ajello (2016), and Bigio (2012).

[3]There are thus both fully liquid government issued assets and partially liquid private claims. Changing the portfolio compositions of the two assets can potentially affect the real economy. More recently, financial intermediations are added and the policy affects the asset compositions held by intermediaries. See, for

This paper provides a novel channel in which public liquidity provision is costly due to distortionary taxation. Therefore, an optimal supply of public liquidity emerges.

In this paper, government debt provides liquidity service and has the "crowding-in" effect, similar to Woodford (1990). This aspect is in contrast with Aiyagari and McGrattan (1998) in which government debt is a perfect substitute to private assets (or capital stock). There, government debt relaxes agents' borrowing constraints but also has the "crowding-out" effect on capital accumulation. Aiyagari and McGrattan (1998) also only focus on steady-state welfare analysis, ignoring transition paths. It is unclear whether the optimal debt-to-GDP ratio is affected by the initial state. The planner's problem in this paper takes fully into account the initial capital stock and transition paths. I also show that under reasonable utility functions the debt-to-GDP ratio is unique and independent of the initial condition.

In Angeletos, Collard, Dellas, and Diba (2013), government bonds also "crowd in" resources as they have a higher level of exogenous *pledgeability* as collateral than capital assets. This paper, however, features endogenous asset saleability (and a bid-ask spread) instead of asset pledgeability. Public liquidity provision can also alter the liquidity of privately issued claims through the endogenous search and matching. Therefore, the portfolio choice between private claims and public debt enters the center stage of the planner's problem.

The rest of the paper is organized as follows. Section 2 presents the model, where the characterization is contained in Section 3. In particular, I derive the "implementability constraint" and set up the Ramsey plan via the so called "primal approach". Section 4 presents analytical and numerical results of how an economy should set taxes and debt. Section 5 gives final remarks including future research prospects.

2 A Neoclassical Economy with Liquidity Frictions

Consider a production economy populated by firms, a continuum of similar households (with measure one), financial intermediaries, and a (benevolent) government. Time is discrete and infinite, and there is no aggregate uncertainty.

Competitive firms rent capital stock K_t and labor L_t from the household to produce final consumption goods. The competitiveness ensures that rental rate r_t and wage rate W_t are equal to marginal products:

$$r_t = A_t F_K(K_{t-1}, L_t) \text{ and } w_t = A_t F_N(K_{t-1}, L_t) \tag{1}$$

where A_t is aggregate productivity and $F(.,.)$ is a standard constant-return-to-scale pro-

example, Gertler and Karadi (2011) and Gertler and Kiyotaki (2010).

duction function. The output can be used for private consumption C_t, investment X_t, and government consumption G_t. The government consumption G_t is financed by proportional taxes on the income from labor and capital and by debt.

The other types of agents are discussed in the following one by one.

2.1 A Household with Heterogeneous Agents

At the start of t, all members in a household *equally divide* the household's financial assets consisting of nominal government bonds and privately-issued financial claims on capital stock. The household instructs its members on the optimal type-specific choices to be carried out after individual types realized.

Then, each member receives a status draw, becoming an entrepreneur (type i) with a probability χ and a worker (type n), otherwise. The members are temporarily separated during the period. Capital is rented to firms, and workers supply labor hours. Each member thus receive capital and labor income from firm production.

Only entrepreneurs have access to investment projects transforming consumption goods into capital stock one-for-one. Financial claims and government bonds are thus traded in exchange for consumption goods to be used for investment. After investment and consumption, members unite again in their respective households, pooling all assets together.

Preferences. Each individual has a preference $u(c, l)$ over consumption c and labor l. u is strictly increasing in consumption, is decreasing in labor, is strictly concave, and satisfies the Inada conditions. The household's preferences on consumption and leisure can be represented by an aggregation:

$$\sum_{t=0}^{\infty} \beta^t \Big\{ \chi u(c_t^i, l_t^i) + (1-\chi) u(c_t^n, l_t^n) \Big\} \tag{2}$$

where $\beta \in (0,1)$ is a discount factor and the superscript i (or n) denote the variable of an entrepreneur (or a worker). Note that the superscripts will be used throughout the paper.

Balance sheet. Households hold b_{t-1} units of nominal and fully liquid government bonds with one period maturity and nominal price level P_t. In addition, physical capital k_{t-1} is owned by households and depreciates to $(1-\delta)k_{t-1}$ at the end of each period t, where $\delta \in (0,1)$. Finally, households can hold private financial assets. All these assets are issued as financial claims to the future return of capital stock. For example, the owner of one unit of claims issued at time $t-1$ is entitled to r_t at t, $(1-\delta)r_{t+1}$ at time $t+1$, $(1-\delta)^2 r_{t+2}$ at time $t+2$, and so on.

Hence, the household owns a portfolio of bonds, private claims issued by other households, and the fraction of their own capital stock which has *not* been issued to others. New claims have the same liquidity as claims already issued, since both new and existing claims would need to be traded on the same financial market introduced later. Therefore, besides liquid assets b_{t-1}, we only need to keep track of net private claims

$$s_{t-1} \equiv \text{claims on others' capital stock} + \text{unissued capital stock}$$

As all assets are equally divided among members, every member has s_{t-1} units of private claims and b_{t-1} units of government bonds. Then, let s_t^j and b_t^j denote the end-of-period total net private claims and bonds for type $j \in \{i, n\}$ members.

Asset accumulation. Only entrepreneurs can transform one unit of consumption goods into one unit of capital goods. Investment x_t at time t will be available as capital stock in period $t+1$: $k_t = (1-\delta)k_{t-1} + x_t$. Importantly, no insurance market exists for idiosyncratic investment opportunities. To finance the investment, an entrepreneur can sell government bonds, issue private claims to the future output from the investment, or sell existing claims (all in exchange for consumption goods). Both new issuance and reselling go through financial intermediaries with a costly search and matching technology, and only an *endogenous* fraction $\phi_t \in [0,1]$ (i.e., *asset saleability*) of new or existing assets can be successfully sold with a price q_t^i. Buyers (who turn out to be the workers) need to pay a price $q_t^n > q_t^i$ because of the search costs.

Household members face two financing constraints. First, no private agent can issue government bonds, i.e.,

$$b_t^j \geq 0 \tag{3}$$

The second constraint relates to the accumulation of private claims. For each group j, the position of net private claims evolves according to

$$s_t^j = (1-\delta)s_t + x_t^j - m_t^j \geq (1-\phi_t)\left[x_t^j + (1-\delta)s_{t-1}\right] \tag{4}$$

where x_t^j is the level of investment, and m_t^j denotes the units of private claims sold on the financial market. Due to search frictions, agents need to retain at least a fraction $(1-\phi_t)$ of their existing private claims and of new investment, limiting the external funding for new investment. Finally, as assets are pooled together at the end of t, we know that

$$s_t = s_t^i + s_t^n \text{ and } b_t = b_t^i + b_t^n \tag{5}$$

Next, I move to agents' flow-of-funds constraints (or budget constraints). Let τ_t^k and τ_t^l be the flat tax rate on the income from capital and labor. Let the nominal return of government bonds be R_{t-1} from $t-1$ to t.

Workers' flow-of-funds constraint. All workers are the same, and the worker group does not invest ($x_t^n = 0$). They accumulate financial assets ($m_t^n < 0$ and $b_t^n > 0$) to implement their household's intertemporal consumption smoothing plan. As a result, neither of their financing constraints is binding. A worker uses labor income, the return on private claims and bonds, to finance consumption (c_t^n) and the end-of-period portfolio of private claims (s_t^n) and bonds (b_t^n):

$$c_t^n + q_t^n s_t^n + \frac{b_t^n}{P_t} = (1-\tau_t^l)w_t l_t^n + \left[(1-\tau_t^k)r_t + q_t^n(1-\delta)\right] s_{t-1} + \frac{R_{t-1}b_{t-1}}{P_t} \tag{6}$$

where private claims are purchased at the price q_t^n.

Entrepreneurs' flow-of-funds constraint. An entrepreneur needs to finance new investment $x_t^i > 0$. She can use return on private claims and bonds, and the issuance (or reselling) of private claims $m_t^i = x_t^i + (1-\delta)s_{t-1} - s_{t-1}^i$ to finance consumption c_t^i, new bonds b_t^i, and physical investment x_t^i:

$$c_t^i + \frac{b_t^i}{P_t} + x_t^i = (1-\tau_t^k)r_t s_{t-1} + \frac{R_{t-1}b_{t-1}}{P_t} + q_t^i\left[x_t^i + (1-\delta)s_{t-1} - s_t\right] \tag{7}$$

where private claims are issued or resold at the price q_t^i. It is worth noting that q_t^i is also equal to Tobin's q: the ratio of the market value of capital to the replacement cost (i.e., unity).

As long as $q_t^i > 1$, entrepreneurs will use all available resources to create new capital. Assuming $q_t^i > 1$ and I will argue later this is the case for the interesting equilibrium, both financing constraints (3) and (4) bind. Intuitively, Because of the equal division of assets in the beginning, entrepreneurs do not have enough resources to finance the first-best investment. Hence, $s_t^i = (1-\phi_t)\left[x_t^i + (1-\delta)s_{t-1}\right]$ according to (4), and investment can be written as $x_t^i = \frac{s_t^i - (1-\phi_t)(1-\delta)s_{t-1}}{1-\phi_t}$. Then, the constraint (7) becomes

$$c_t^i + q_t^r s_t^i = \frac{R_{t-1}b_{t-1}}{P_t} + \left[(1-\tau_t^k)r_t + (1-\delta)\right] s_{t-1} \tag{8}$$

$$\text{where} \quad q_t^r \equiv \frac{1-\phi_t q_t^i}{1-\phi_t} \leq 1 \tag{9}$$

The right-hand side of (8) is entrepreneurs' total net-worth. On the left-hand side, s_t^i is valued at q_t^r which is the *effective* replacement cost of capital. To see this, notice that for every unit of new investment, a ϕ_t fraction is issued at the price q_t^i; entrepreneurs need to make a "down-payment" $(1-\phi_t q_t^i)$ and retain a fraction $(1-\phi_t)$ as inside equity claims. The inside equity claims, s_t^i is thus valued at $q_t^r \equiv \frac{1-\phi_t q_t^i}{1-\phi_t}$. The lower q_t^r is, the larger s_t^i is and they can bring more private claims back to the household.

Once we know s_t^i from (8), individual investment $x_t^i = \frac{s_t^i-(1-\phi_t)(1-\delta)s_{t-1}}{1-\phi_t}$ and aggregate investment X_t can be backed out as

$$x_t^i = \frac{\left[r_t + (1-\delta)\phi_t q_t^i\right] s_{t-1} + \frac{R_{t-1}b_{t-1}}{P_t} - c_t^i}{1-\phi_t q_t^i} \text{ and } X_t = \chi x_t^i \tag{10}$$

Equation (10) says that entrepreneurs' *liquid net-worth*, including return from private claims and bonds, and the saleable part of existing claims $(1-\delta)\phi_t q_t^i \chi S_t$ can be "leveraged" with a factor $(1-\phi_t q_t^i)^{-1}$ to invest in new capital. A fall of asset saleability ϕ_t or asset price q_t^i reduces investment. Investment also drops when the (real) public liquidity $R_{t-1}b_{t-1}/P_t$ falls.

2.2 Financial Markets

Financial markets open in which entrepreneurs offer financial claims for sale and workers purchase these claims through financial intermediaries, which implement a costly search and matching technology. Search frictions imply that private financial claims are only partially liquid. In contrast, government bonds are fully liquid as they can be traded on a frictionless spot market.

Search and matching. There are capital sub-markets $m = 1, 2, ...$ with free entry of financial intermediaries. As we shall see, the number of sub-markets is not important for our analysis. On each market, workers post V_t^m units of buy quotes, and entrepreneurs post U_t^m units of sell offers backed by capital stock. As will be clear, intermediaries screen submarkets for valuable projects to invest in.

In order to match their buy quotes with suitable sell quotes in a particular sub-market m, intermediaries operate a matching technology that determines the number of matched claims M_t^m

$$M_t^m = M(U_t^m, V_t^m) = \xi \left(U_t^m\right)^{\eta} \left(V_t^m\right)^{1-\eta}$$

where ξ is matching efficiency and η is the matching elasticity. The matching technology endogenizes the probabilities of filling a sell quote, $\phi_t^m \equiv M(U_t^m, V_t^m)/U_t^m$, and, conversely, of filling a buy quote, $f_t^m \equiv M(U_t^m, V_t^m)/V_t^m$. Therefore,

$$f_t^m = \xi^{\frac{1}{1-\eta}} \left(\phi_t^m\right)^{\frac{\eta}{\eta-1}} \tag{11}$$

Then, $\theta_t^m \equiv V_t^m/U_t^m = \xi^{\frac{1}{\eta-1}} (\phi_t^m)^{\frac{1}{1-\eta}}$ is the *search intensity* of sub-market m and positively co-moves with ϕ_t^m. To maximize external funding, entrepreneurs post quotes amounting to $U_t^m = I_t^i + (1-\delta)\chi S_t$, of which $\phi_t^m U_t^m$ can be sold. ϕ_t^m, indeed, captures asset saleability.

Financial Intermediation. There is an inter financial-intermediaries market. Financial intermediaries can sell one unit of claims - acquired from entrepreneurs at the price $q_t^{i,m}$ - to other intermediaries on the same capital submarket m at the price q_t^m. Alternatively, intermediaries can sell claims trading at price q_t^m on sub-market m to workers at price $q_t^{n,m}$.

Both buy quotes and sell quotes should be committed, and the intermediaries need to pay κ units of consumption goods per quote to monitor the delivery from either the buyers (with consumption goods) and the sellers (with financial claims backed by capital). Since only a fraction f_t^m of buy quotes is matched, the cost of selling one unit of claims for an intermediary to another intermediary is κ/f_t^m. Because of the competitive environment, the following zero-profit condition must then hold in each sub-market:

$$\frac{\kappa}{f_t^m} = q_t^m - q_t^{i,m} \tag{12}$$

Since only a fraction ϕ_t^m of sell quotes is matched, the cost of selling one unit of claims to workers is κ/ϕ_t^m. We thus have another zero-profit condition:

$$\frac{\kappa}{\phi_t^m} = q_t^{n,m} - q_t^m \tag{13}$$

In reality, the asset position may not be cleared instantaneously in the hands of financial intermediaries. One could interpret that intermediaries charge the buyers and sellers compensation for holding assets. The total costs for one unit of assets sold from sellers to buyers amount to $\kappa\left(\frac{1}{\phi_t^m} + \frac{1}{f_t^m}\right)$. Importantly, κ represents the financial frictions in the model. Without κ (or when $\kappa = 0$), buyers and sellers can post whatever amount of quotes to achieve their first-best outcomes.

Asset price. In light of the two zero-profit conditions, intermediaries are indifferent between all submarkets. In addition, workers go to the submarket with the lowest $q_t^{n,m}$, which is

$$q_t^n = q_t^i + \kappa_t\left(\frac{1}{\phi_t} + \frac{1}{f_t}\right) \tag{14}$$

We can thus omit the superscript m. Given these constraints, each submarket is characterized by its saleability-sell-price combination (ϕ, q^i). Accordingly, entrepreneurs choose the submarket in which to post their sell offers, which minimizes the effective replacement cost q^r, subject to the zero-profit condition and the relationship between f and ϕ:

$$\min_{\{0\leq\phi_t\leq1,\quad q_t^i\geq1\}} \quad q_t^r = \frac{1-\phi_t q_t^i}{1-\phi_t} \tag{15}$$

subject to (12) and (11). This maximizes the end-of-period s_t^i, according to the entrepreneurs' flow-of-funds constraint (8).

2.3 The Government

The government sets tax rates on labor (τ_t^l) and capital income (τ_t^k) and nominal returns (R_t) for government debt to finance the exogenous sequence of government expenditures. Let $K_{t-1} = k_{t-1}$, $B_{t-1} = b_{t-1}$, and $L_t = \chi l_t^n$ be the aggregate capital, bonds, and labor hours. The government's budget constraint is then

$$G_t + \frac{B_t}{P_t} = \tau_t^k r_t K_{t-1} + \tau_t^l w_t L_t + \frac{R_{t-1} B_{t-1}}{P_t} \tag{16}$$

The initial capital stock K_{-1}, nominal debt B_{-1}, and nominal interest rate R_{-1}, are given.

3 Competitive and Ramsey Equilibrium

Throughout, aggregate productivity A_t, government consumption G_t, and cost of search and matching κ_t are exogenously specified and *deterministic*. Again, I focus on the type of equilibrium in which entrepreneurs are financing constrained (which will be verified) and both private claims and government bonds circulate.

In the following, I first define the competitive equilibrium and the Ramsey equilibrium. Then, I characterize the competitive equilibrium, which will be used in the policy problem faced by the benevolent government in the Ramsey equilibrium. Finally, I show how to set up a Ramsey problem via the primal approach.

3.1 Equilibrium Definition

Let $\xi_t = (c_t^i, s_t^i, c_t^n, s_t^n, l_t^n, b_t^n)$ denote an allocation and $\zeta_t = (\tau_t^k, \tau_t^l, B_t, R_t)$ denote the government policy at time t. Then, $\xi = \{\xi_t\}$ and $\zeta = \{\zeta_t\}$ denote the infinite sequence of allocations and policies. I also let $(r_t, w_t, q_t^i, q_t^n, \theta_t, P_t)$ denote a price system

Definition 1: Given the deterministic and exogenous aggregate state (A_t, κ_t, G_t), a *competitive equilibrium (CE)* is an allocation ξ, a policy ζ, and a price system $(r_t, w_t, q_t^i, q_t^n, \theta_t, P_t)$ such that

1. given the policy and the price system, the allocation maximizes the household's utility (2) subject to the flow-of-funds constraints (6) and (8), and financing constraints (3) and (4);

2. (r_t,w_t) satisfies (1); q_t^n satisfies (14);

3. the resulting θ_t, q_t^i) maximize (15) subject to (14) and (11);

4. P_t satisfies the government budget constraint (16), with $K_t = \chi s_t^i + (1-\chi) s_t^n$, $B_t = b_t^n$, and $L_t = (1-\chi) l_t^n$ in equilibrium.

From the above equilibrium definition, Walras law implies that the feasibility condition (the social resource constraint) is satisfied. To see this, multiply a worker's budget constraint (6) by $(1-\chi)$ and an entrepreneur's budget constraint (8) by χ, add the government budget constraint (16), and obtain:

$$\begin{aligned} &\chi c_t^i + (1-\chi) c_t^n + G_t + K_t + \kappa_t (1+\theta_t) \left[K_t - (1-\delta) K_{t-1} + (1-\delta)\chi K_{t-1}\right] \\ =& A_t F\left(K_{t-1}, N_t\right) + (1-\delta) K_{t-1} \end{aligned} \tag{17}$$

where the total consumption is equal $\chi c_t^i + (1-\chi) c_t^n$. In (17), the total amount of asset transaction is $\phi_t \left[K_t - (1-\delta) K_{t-1} + (1-\delta)\chi K_{t-1}\right]$ which includes a ϕ_t fraction of aggregate investment $K_t - (1-\delta) K_{t-1}$ and a ϕ_t fraction of existing financial claims $(1-\delta)\chi K_{t-1}$ held by entrepreneurs. Given that the cost for a transaction is $\kappa_t (d\phi_t^{-1} + f_t^{-1})$, therefore we know the total transaction cost is $\kappa_t (d\phi_t^{-1} + f_t^{-1})$ times the total amount of asset transaction which becomes $\kappa_t (d + \theta_t) \left[K_t - (1-\delta) K_{t-1} + (1-\delta)\chi K_{t-1}\right]$. Therefore, with other equilibrium conditions in presence, we can either use the government budget constraint or the social resource constraint since one implies the other.

I now move to the policy problem faced by the benevolent government. Since I am interested in the Ramsey problem, I assume that there is enough commitment technology through which the government can commit itself to a particular sequence of policies from period 0. That is, the government chooses a policy ζ at time 0 and then all agents choose their allocations. Therefore, allocation rule $\xi = \xi(\zeta)$ is a mapping that maps policies ζ into allocations; price rules are also sequence of functions $r(\zeta)$, $w(\zeta)$, $q^i(\zeta)$, $q^n(\zeta)$, $\theta(\zeta)$, and $P(\zeta)$, that map policies ζ into price systems.

The benevolent government needs to predict the allocations and prices for a given policy ζ. In addition, there should be restrictions on the initial capital tax rate τ_0^k. The government might want to set τ_0^k as high as possible due the fact that K_{-1} is given. Then, the Ramsey problem becomes trivial. Without the loss of generality, I therefore set initial tax rate $\tau_0^k = 0$. (One can set τ_0^k equal a small number and all derivations below still work.)

Definition 2: A *Ramsey equilibrium (RE)* is a policy ζ, an allocation rule $\xi(\zeta)$, and price rules $r(\zeta)$, $w(\zeta)$, $q^i(\zeta)$, $q^n(\zeta)$, and $P(\zeta)$, that satisfy

1. the policy ζ maximizes the household's utility

$$\sum_{t=0}^{\infty} \beta^t \Big\{ \chi u(c_t^i(\zeta), 0) + (1-\chi) u(c_t^n(\zeta), l_t^n(\zeta)) \Big\} \text{ s.t. } (16)$$

   with $K_t = \chi s_t^i + (1-\chi) s_t^n$, $B_t = b_t^n$, $L_t = (1-\chi) l_t^n$, and prices given by the price rules;

2. for every policy $\tilde{\zeta}$, the allocation $\xi(\tilde{\zeta})$ and price rules $r(\tilde{\zeta})$, $w(\tilde{\zeta})$, $q^i(\tilde{\zeta})$, $q^n(\tilde{\zeta})$, and $P(\tilde{\zeta})$ generate a competitive equilibrium.

3.2 Characterizing a Ramsey Equilibrium

To characterize a Ramsey equilibrium, I need to first characterize a competitive equilibrium, given government policies. In particular, I derive an implementability condition faced by the government. Such implementability condition summarizes decisions made by the private agents. Then, I show how to set up the optimal policy problem with commitment, i.e., a Ramsey problem.

3.2.1 Competitive Equilibrium

First, I characterize the financial market. The asset price and asset saleability is linked through the search-and-matching process:

Proposition 1. *Assuming an interior solution, one can express asset price q_t^i as a function of search costs κ_t and search intensity θ_t*

$$q_t^i = 1 + \frac{\kappa_t \eta}{(1-\eta)} \frac{(1-\phi(\theta_t))}{f(\theta_t)} \tag{18}$$

Asset price q_t^i fall with asset saleability ϕ_t if the search intensity $\theta_t < (\eta/\xi)^{\frac{1}{1-\eta}}$.

Proof. See the Appendix. □

Notice that $q_t^i > 1$, if $\kappa_t > 0$ and $\phi_t \neq 1$. Intuitively, private claims cannot be too liquid, otherwise q^i becomes one and entrepreneurs are not financing constrained. (That is, the internal cost of investment is the same as the external cost of investment.) Importantly, one can see that asset saleability ϕ_t directly affects asset price q_t^i for private claims. This fact implies that individual entrepreneurs consider the asset trading's impact on asset prices. They choose optimally the pair (ϕ_t, q_t^i). While for government bonds which are fully liquid, individuals do not consider the trading's impact on the bond price.

Notice further that asset price q_t^i *can* fall with asset saleability ϕ_t, when search intensity is small or the market is tight. This is because the fall of ϕ_t is driven by the fall of search intensity θ_t. When θ_t falls, demand (represented by V_t) always fall more than the supply (represented by U_t). When θ_t is small, demand is low and asset price is sensitive to demand movements. As a result, entrepreneurs can be more financing constrained (ϕ_t is smaller) in an environment of falling asset price q_t^i. A joint fall in ϕ_t and q_t^i further tightens entrepreneurs' financing constraints, as shown in the investment equation (10).[4]

Second, I derive the household's decision rules. The following first-order necessary conditions are also sufficient due to the concavity of the objective function. For notation simplicity, I use $u_{c,t}^j$ and $u_{l,t}^j$ to denote the marginal utilities of consumption and labor hours for type j agents at time t. The first-order condition for labor is

$$(1-\tau_t^l)w_t u_{c,t}^n = -u_{l,t}^n \tag{19}$$

which implies a standard intra-period tradeoff: the marginal gain from working an extra hour which brings more consumption needs to be equalized with the marginal cost from disutility from working.

The first-order conditions for s_t^i and s_t^n are

$$u_{c,t}^i = \frac{q_t^n}{q_t^r} u_{c,t}^n = \rho_t u_{c,t}^n \tag{20}$$

$$\frac{\beta u_{c,t+1}^n}{u_{c,t}^n}\left[\chi\rho_{t+1}\underbrace{\left(\frac{(1-\tau_{t+1}^k)r_{t+1}+(1-\delta)}{q_t^n}\right)}_{r_{t+1}^{ni}}+(1-\chi)\underbrace{\left(\frac{(1-\tau_{t+1}^k)r_{t+1}+(1-\delta)q_{t+1}^n}{q_t^n}\right)}_{r_{t+1}^{nn}}\right]=1, \tag{21}$$

In (20), $\rho_t \equiv q_t^n/q_t^r$ measures the degree of financial frictions. In a standard neoclassical model, agent can share risks together. Then, $q_t^n = q_t^i = 1$ and therefore $q_t^r = 1$. This means that $\rho_t = 1$ and from (20), everyone indeed consumes the same. In the current model with financial frictions, however, since u is concave and $\rho_t > 1$, (20) tells us that entrepreneurs consume less than workers because they need to invest in physical capital. For notation

[4] As pointed out by Shi (2015), when ϕ_t is exogenous, a reduction of ϕ_t always pushes up asset price. This is mainly due to the fact when ϕ_t is exogenous, one has a constraint on the supply of new and old assets. A fall in ϕ_t can only put higher pressure on q_t^i. Introducing search frictions avoids this issue, as ϕ_t reflects both asset demand and supply. See Cui and Radde (2016), Cui and Radde (2015), and Cui (2016) for more detailed analysis.

convenience, we know that there exists a function $g = (u_c^i)^{-1}$ such that

$$c_t^i = g(\rho_t u_{c,t}^n) \tag{22}$$

In (21), $\beta u_{c,t+1}^n / u_{c,t}^n$ is the stochastic discount factor of a worker. The two returns r_{t+1}^{ni} and r_{t+1}^{nn} resemble the workers' internal returns on private claims, if she is an entrepreneur or worker at date $t+1$, respectively. From the flow-of-funds constraints at date $t+1$, an entrepreneur's valuation of one unit of private claims is $(1-\delta)$, while the workers' valuation is $(1-\delta)q_{t+1}^n$. Then, $\chi\rho_{+1} r_{+1}^{ni} + (1-\chi) r_{+1}^{nn}$ is the household's internal return with the adjustment of ρ_{t+1}. Notice that ρ_{t+1} represents liquidity risks, or the marginal utility of consumption of an entrepreneur relative to that of a worker. I thus interpret equation (21) as a *liquidity-adjusted asset pricing formula* for private claims.

Following similar steps, I derive another first-order condition for government bonds b_t^n

$$\frac{\beta u_{c,t+1}^n}{u_{c,t}^n} \frac{R_t \left(\chi \rho_{t+1} + 1 - \chi\right)}{P_{t+1}/P_t} = 1 \tag{23}$$

where similarly $\frac{P_t R(\chi\rho_{t+1}+1-\chi)}{P_{t+1}}$ is the household's internal return on government bonds. Unlike private claims, the bond return for both entrepreneurs and workers is $P_t R_t / P_{t+1}$ because government bonds are liquid assets. With the adjustment of ρ_{t+1}, I again interpret equation (23) as a liquidity-adjusted asset pricing formula for government bonds.

It is helpful to define liquidity premium as the return difference between private claims and government bonds, i.e.,

$$\Delta_{t+1} \equiv \chi r_{t+1}^{ni} + (1-\chi) r_{t+1}^{nn} - P_t R_t / P_{t+1}$$

By using the two first-order conditions just derived (i.e., (21) and (23)), we know that Δ_{t+1} is positive if $\rho_{t+1} > 1$. That is, when $\rho_{t+1} > 1$, agents who become entrepreneurs tomorrow will be financing constrained. These agents hold government bonds as precautionary savings, driving down the real return of government bonds.

Proposition 2. *Assuming $\kappa_t > 0$ and both private claims and government bonds co-exist. Then, $\rho_{t+1} > 1$ and the premium of return on private claims relative to return from government bonds is positive, i.e., $\Delta_{t+1} > 0$. The higher is ρ_{t+1}, the larger is the premium Δ_{t+1}.*

Proof. See the Appendix. □

The intuition behind the proposition is the following. In such deterministic framework, in order to let workers to hold government bonds with a lower return than private claims, government bonds need to be special in some aspects. Government bonds are special because they do not need to be searched, and they are especially valuable when an entrepreneur is financing constrained, i.e., when $\rho_t > 1$. Workers expect that in the future they may become entrepreneurs and thus financing constrained. Therefore, they are willing to hold liquid government bonds with a lower return today.

Lastly, I link the financial market tightness θ and the household's risk-sharing ρ. Since q_t^n and q_t^i can be expressed as functions of search intensity θ (given exogenous search costs κ_t), $\rho_t = q_t^n/q_t^r$ is also a function of θ_t.

Proposition 3. *The relationship between risk-sharing ρ_t and search market intensity θ_t is*

$$\rho_t = \rho(\theta_t) \equiv 1 + \frac{\theta_t + (1-\eta)}{\xi\theta_t^{1-\eta}\left[(1-\eta)\kappa_t^{-1} - \eta\theta_t\right]}$$

$\partial\rho/\partial\theta < 0$ *if and only if*

$$0 < \theta < \frac{-\left(\kappa^{-1}+2-\eta\right) + \sqrt{\left(\kappa^{-1}+2-\eta\right)^2 + \frac{4(1-\eta)^2}{\kappa\eta}}}{2}.$$

$\partial\rho/\partial\theta > 0$ *if and only if*

$$\frac{-\left(\kappa^{-1}+2-\eta\right) + \sqrt{\left(\kappa^{-1}+2-\eta\right)^2 + \frac{4(1-\eta)^2}{\kappa\eta}}}{2} < \theta < \frac{1-\eta}{\kappa\eta}.$$

Proof. See the Appendix. □

When θ rises, it could affect the gap $\rho_t = q_t^n/q_t^r$ differently, depending on where the original θ is.

When θ is small, the market is tight and entrepreneurs' financing constraint is also tight. A higher θ implies that entrepreneurs can obtain better outside funding situation from the financial market with a higher asset price q_t^i. This reduces the consumption gap between entrepreneurs and workers, i.e., $\partial\rho/\partial\theta < 0$.

When θ is high, the market is already not tight. A further higher θ implies that asset price q_t^i needs to fall to reflect the more supply of assets. This increases the consumption gap between entrepreneurs and workers, i.e., $\partial\rho/\partial\theta > 0$.

3.2.2 The Implementability Condition and the Ramsey Problem

To prepare for the Ramsey problem, I show that a competitive equilibrium can be characterized by two simple conditions. One is the social resources constraint (17), and the other is the implementability condition:

$$\sum_{t=0}^{\infty} \beta^t \left[u^n_{c,t} \left[\chi \rho_t g \left(\rho_t u^n_{c,t} \right) + (1-\chi) c^n_t \right] + u^n_{l,t} L_t \right] - H = 0 \tag{24}$$

where H is given by

$$\begin{aligned} H(c^n_0, \rho_0, K_0; K_{-1}) \equiv\; & u^n_{c,0} \left(\chi \rho_0 + 1 - \chi \right) \left[\left(\frac{1 - \phi(\rho_0) q^i(\rho_0)}{\chi} \right) K_0 + g\left(\rho_0 u^n_{c,0} \right) \right] \\ & + u^n_{c,0} (1-\chi) \left[q^n(\rho_0) - 1 \right] (1-\delta) K_{-1} \end{aligned} \tag{25}$$

The first step is to show that any competitive equilibrium leads to the two conditions above. The second step is to show that with the allocations that satisfy the two conditions, one can construct government policies committed at time 0 that generates a competitive equilibrium.

The first step can be shown through the following proposition.

Proposition 4. *Given an initial capital tax rate τ^k_0 and a nominal interest rate R_{-1}, the allocations in a competitive equilibrium satisfy (17) and (24).*

Proof. See the Appendix. □

The implementability condition depends on the allocations of consumption C_t, labor supply L_t, and ρ_t. However, the condition is independent of capital stock K_{t-1} and government bonds B_{t-1}, except for the initial capital stocks K_{-1}. Households' portfolio choices and endogenous asset prices have thus been substituted out.

It may seems at first that initial capital tax τ^k_0 and initial nominal interest rate R_{-1} is not used in the implementability condition. But they are important to determine the initial price level P_0. They need to be restricted such that P_0 is a finite positive number. We will see this point in the following illustration.

In the second step, I should show the opposite that any CE allocations can be constructed by certain government policies.

Suppose that we have allocations and period 0 policies that satisfy (17) and (24). I show that one can generate the competitive equilibrium as follows. Since we have all sequence of $\{K_t\}$ and $\{L_t\}$, we know the rental rate $\{r_t\}$ and wage rate $\{w_t\}$ from (1). The labor tax rate is given by (19) where $(1-\tau^l_t) = -u^n_{l,t}/(w_t u^n_{c,t})$. Capital tax rate is given by the first-order condition for private claims s^n_t in (21).

I am left to determine the debt levels, the nominal interest rates, and the price levels. Not surprisingly, there is one degree of freedom between choosing a nominal interest rate and a nominal price level. To see this more clearly, in any period $t \geq 0$, R_{t-1} and B_{t-1} are predetermined. Then, the entrepreneur's budget constraint (8) determines the price level P_t. In particular, to reflect what was discussed before, P_0 is determined by having pre-specified τ_0^k and R_{-1}. With the knowledge of P_t, one can back out B_t from the combined budget constraint (45), leaving R_t indetermined. Therefore, the government can freely choose nominal interest rate. For example, any feedback interest rate rule $R_t(P_t)$ is enough.

For notation simplicity, I define aggregate consumption as $C_t = \chi c_t^i + (1-\chi)c_t^n$. Because $c_t^i = g(\rho_t u_{c,t}^n)$ and u is concave, we can use two invariant functions h^n and h^c that map aggregate consumption and the degree of risk-sharing into individual consumption:

$$c_t^n = h^n(C_t, \rho_t) \text{ and } c_t^i = h^i(C_t, \rho_t)$$

Then, let $\Phi \geq 0$ be the Lagrangian multiplier associated with the implementability condition and define the planner's per-period social utility as

$$\begin{aligned} J\left(C_t, L_t, \rho_t; \Phi\right) \equiv\ & \chi u\left(h^i(C_t, \rho_t), 0\right) + (1-\chi)\, u\left(h^n(C_t, \rho_t), \frac{L_t}{1-\chi}\right) + \Phi u_l\left(h^n(C_t, \rho_t), \frac{L_t}{1-\chi}\right) L_t \\ +\ & \Phi u_c\left(h^n(C_t, \rho_t), \frac{L_t}{1-\chi}\right)\left[\chi \rho_t h^i(C_t, \rho_t) + (1-\chi) h^n(C_t, \rho_t)\right] \end{aligned}$$

The Ramsey planner's problem can be thus written as two separate sub problems

Problem 1: Given K_{-1} and $\Phi \geq 0$, the social planner solves the following problem

$$\max_{\{C_t, L_t, \theta_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t J\left(C_t, L_t, \rho(\theta_t); \Phi\right) - \Phi H$$

$$\begin{aligned} s.t. C_t + G_t + K_t = &\ A_t F\left(K_{t-1}, L_t\right) + (1-\delta) K_{t-1} \\ & - \kappa_t \left(1+\theta_t\right)\left[K_t - (1-\delta)K_{t-1} + (1-\delta)\chi K_{t-1}\right] \quad \forall t \geq 0. \end{aligned}$$

where

$$\begin{aligned} H = &\ u_c\left(h^n(C_0, \rho_0(\theta_0)), \frac{L_0}{1-\chi}\right)\left(\chi \rho_0(\theta_0) + 1 - \chi\right)\left[\left(\frac{1-\phi_0(\theta_0) q_0^i(\theta_0)}{\chi}\right) K_0 + h^i(C_0, \rho_0(\theta_0))\right] \\ & + u_c\left(h^n(C_0, \rho_0(\theta_0)), \frac{L_0}{1-\chi}\right)(1-\chi)\left(q_0^n(\theta_0) - 1\right)(1-\delta) K_{-1} \end{aligned}$$

Problem 2: The planner chooses a Φ, and the induced allocations obtained in Problem 1 should satisfy the implementability condition (24).

3.2.3 Characterizing the Ramsey Allocation

Now, I characterize the planner's solution. I choose to eliminate C_t by using the social resources constraint so that the planner only chooses labor, search intensity, and capital $\{L_t, \theta_t, K_t\}$. Assuming interior solutions, i.e., $\rho > 1$, we know that the first-order conditions at time $t \geq 1$ is different from $t = 0$ due to the presence of ΦH in the objective. I first illustrate the case when $t \geq 1$:

$$J_{L,t} = -J_{C,t} A_t F_L(K_{t-1}, L_t) \tag{26}$$

$$J_{\rho,t} \frac{\partial \rho(\theta_t)}{\partial \theta_t} = J_{C,t} \kappa_t \left[K_t - (1-\delta)(1-\chi)K_{t-1}\right] \tag{27}$$

$$\begin{aligned} J_{C,t}\left[1 + \kappa_t\left[1 + \theta_t\right]\right] = \beta J_{C,t+1} \Big\{ & A_{t+1} F_K(K_t, L_{t+1}) \\ & + \left[1 + \kappa_{t+1}\left[1 + \theta_{t+1}\right](1-\chi)\right](1-\delta) \Big\} \end{aligned} \tag{28}$$

(26) is the planner's version of intertemporal optimization between leisure and consumption.

(27) is a trade-off between improving risk-sharing (inequality) and the cost of doing so. To see this, suppose $J_C > 0$ and $J_\rho < 0$ (this is true for a class of utility functions commonly used in macro as shown later). This is intuitive, since more aggregate consumption is preferred and ρ measures inequality. When ρ is higher, the gap between the consumption of workers and that of entrepreneurs are larger, which is disliked by the planner. Therefore, the planner always prefer a smaller ρ. When ρ is smaller, this means that the search intensity θ is higher as $\partial \theta_t / \partial \rho_t < 0$. That is, when the entrepreneurs are placed in a more liquid financial market, they can get better outside financing and therefore have more consumption. Yet, a more liquid financial market does not come for free. The cost rises by $\kappa_t \left[K_t - (1-\delta)(1-\chi)K_{t-1}\right]$, when θ increases marginally (for a given amount of assets $K_t - (1-\delta)(1-\chi)K_{t-1}$ to be through transaction in the financial market).

(28) represents the planner's intertemporal optimization on capital, taking into account the social cost in the financial market.

For $t = 0$, all first-order conditions are similar to those when $t \geq 1$, except that the planner needs to take into account the initial period constraint, reflected by the Lagrangian multiplier Φ:

$$J_{L,t} - \Phi H_L = -J_{C,t} A_t F_N(K_{t-1}, N_t) \tag{29}$$

$$[J_{\rho,0} - \Phi H_\rho] \frac{\partial \rho(\theta_0)}{\partial \theta_0} = J_{C,0} \kappa_0 [K_0 - (1-\delta)(1-\chi) K_{-1}] \tag{30}$$

$$\begin{aligned}(J_{C,0} - \Phi H_C)[1 + \kappa_0 [1 + \theta_0(\rho_0)]] - \Phi H_K = \beta J_{C,1} \Big\{ & A_1 F_K(K_0, N_1) \\ & + [1 + \kappa_1 [1 + \theta_1(\rho_1)](1-\chi)](1-\delta) \Big\}\end{aligned} \tag{31}$$

The initial period is important for the Ramsey allocation. This point can be illustrated by how one could solve the Ramsey allocation numerically through a recursive method. To see this, first, guess a Lagrangian multiplier $\Phi \geq 0$ and solve for the steady state (for example, by assuming that for a large enough T the economy is at the steady state) by using the feasibility condition and (26)-(28), with $C_t = C_{t-1} = \bar{C}$, $L_t = L_{t-1} = \bar{L}$, $K_t = K_{t-1} = \bar{K}$, and $\rho_t = \rho_{t-1} = \bar{\rho}$. Second, solve for the whole path of $\{C_t,\ L_t,\ K_t,\ \rho_t\}_{t=0}^T$ by using the feasibility condition and all first-order conditions (26)-(31). Finally, check whether the allocations obtained satisfy the implementability conditions. If not, adjust Φ and redo the process.

4 Risk-sharing, Capital Tax, and Government Debt

In this section, I derive some useful analytical results of the Ramsey problem under a class of utility functions that are commonly used. Then, I perform a numerical exercise to illustrate the quantitative implication of long-run level of government debt and the capital tax rates.

4.1 A Useful Class of Utility Functions

In order to gain analytical insight, I find it useful to look at a class of utility functions that are usually calibrated in macro models

$$u(c, l) = \frac{c^{1-\sigma} - 1}{1 - \sigma} - v(l) \tag{32}$$

with some increasing function $v(l)$ and $v(0) = 0$.

This class of functions means that consumption and labor hours are separable, and the utility of consumption has a constant relative risk aversion (CRRA) coefficient σ, or a constant elasticity of intertemporal substitution (EIS) $1/\sigma$. Further more, $J_{C,t}$ and $J_{\rho,t}$ can be significantly simplified as the following

Lemma 1. *For the class of utility function given by* (32)*, we have*

$$c_t^i = h^i(C_t, \rho_t) = \frac{C_t}{\chi + (1-\chi)\rho_t^{1/\sigma}}, \quad c_t^n = h^n(C_t, \rho_t) = \frac{\rho_t^{1/\sigma} C_t}{\chi + (1-\chi)\rho_t^{1/\sigma}}$$

$$J_{C,t} = \frac{\chi + (1-\chi)\rho_t^{1/\sigma - 1}}{\left[\chi + (1-\chi)\rho_t^{1/\sigma}\right]^{1-\sigma}} \left[1 + (1-\sigma)\Phi\right] C_t^{-\sigma} \tag{33}$$

$$J_{\rho,t} = \frac{\chi(1-\chi)\left[\rho_t^{-1} - 1\right]\rho_t^{1/\sigma - 1}}{\sigma\left[\chi + (1-\chi)\rho_t^{1/\sigma}\right]^{2-\sigma}} \left[1 + (1-\sigma)\Phi\right] C_t^{1-\sigma} \tag{34}$$

Proof. See the Appendix. □

In each period $(c_t^i)^{-\sigma} = \rho_t (c_t^n)^{-\sigma}$, and together with the fact that $\chi c_t^i + (1-\chi)c_t^n = C_t$ we know that individual consumption is a proportion of aggregate consumption as shown by $h^i(C_t, \rho_t)$ and $h^n(C_t, \rho_t)$. Further, given aggregate consumption C_t and risk-sharing ρ_t, the tightness of the implementability constraint, Φ, is linear in the planner's marginal value with respect to C and ρ. Such simplification helps understand the Ramsey allocation better.

Finally, given the simplified $J_{C,t}$ and $J_{\rho,t}$, we know that $J_{C,t}/J_{\rho,t} < 0$. According to (27), one needs that $\partial\rho(\theta)/\partial\theta < 0$ in order to have an interior solution (which corresponds to an interesting equilibrium). I therefore impose this assumption. That is, according to (3), θ is small enough. In fact, when one imposes a small θ in equilibrium, it is equivalent to impose a low real interest rate. This is because when θ is small, ρ is large and real interest rate is small according to the Euler equation for bonds (23). We will come back to this point soon.

4.2 Liquidity Premium and Capital Taxation

To see the effect of capital tax on liquidity premium and risk sharing, I look at the capital's first-order condition in the planner's problem (28). Using the class of utility functions, we know that τ_{t+1}^k from period $t \geq 1$ onward only depends on risk-sharing ρ_t and ρ_{t+1}.

Proposition 5. *For utility functions of the form* (32)*, capital tax* τ_{t+1}^k *only depends on the risk-sharing* ρ_t *and* ρ_{t+1} *for any period* $t \geq 1$*.*

Proof. Using the form of (33), we know that

$$\frac{J_{C,t+1}}{J_{C,t}} = \frac{u^n_{c,t+1}}{u_{c,t}} \frac{\chi\rho_{t+1} + (1-\chi)\rho_{t+1}^{1/\sigma}}{\chi\rho_t + (1-\chi)\rho_t^{1/\sigma}} \frac{\chi + (1-\chi)\rho_{t+1}^{1/\sigma}}{\chi + (1-\chi)\rho_t^{1/\sigma}}$$

Next, using this relationship in the planner's first-order condition of capital (28) and the household's first-order condition (23), I substitute out $F_K(K_t, N_{t+1})$ and know that the capital tax rate τ^k_{t+1} is only a function of ρ_t and ρ_{t+1}. □

One implication of the above result is that τ^k_{t+1} moves with the degree of risk-sharing ρ_t. If ρ_t immediately jumps to the steady-state level, so does τ^k_{t+1}, regardless of the capital stock level K_t. This result echoes the previous discussion on capital tax effect on redistribution. Further, since liquidity premium depends on the degree of risk-sharing, we thus know that capital tax is mainly used for achieving the optimal liquidity premium. But one should realize how complex the capital tax effect on liquidity premium is.

First, capital tax has distortionary effect on capital accumulation, which interacts with liquidity premium (recall the two Euler equations). Second, private agents save too much on government bonds due to precautionary motives. The externality leads to a too high level of liquidity premium (or, a too low level of real interest rate).[5] A positive level of capital tax thus corrects the externality and "crowds in" capital accumulation and wealth, mitigating liquidity frictions. Third, capital tax discourages saving in capital and encourages saving in government bonds, raising liquidity premium and putting downward pressure on real interest rate. The government might thus want to impose a capital tax to reduce the cost of financing expenditures.

We shall see the relationship between capital tax and risk-sharing more clearly in a steady state. If the exogenous κ_t, A_t, and G_t converge to a constant, the economy converges to a steady state. Then, $J_{C,t}$ is a constant, and I can compute the planner's intertemporal choice from equations (28)

$$1 + \kappa(1+\theta(\rho)) = \beta[F_K(K,L) + [1 + \kappa\,[1+\theta(\rho)]\,(1-\chi)]\,(1-\delta)] \tag{35}$$

From the Euler equation of the private agents (21), we know $u_{c,t}$ is also a constant. Therefore,

$$q^n(\rho) = \beta[(\chi\rho + 1 - \chi)(1-\tau^k)r + [\chi\rho + (1-\chi)q^n(\rho)]\,(1-\delta)] \tag{36}$$

Comparing (35) and (36), we immediately see that in a steady state, the optimal tax rate

[5] One should note that accumulating government bonds alone does not create wealth in the aggregate, though it can help to reduce liquidity shock for individuals.

on capital crucially depends on κ. When $\kappa > 0$, τ^k should not be zero in general.

Proposition 6. *When $\kappa > 0$, the capital tax rate in the long run is uniquely determined. When $\kappa = 0$, the capital tax rate in the long run is zero.*

Proof. When $\kappa > 0$, we can substitute out r by using $r = F_K(K, N)$ in (35) and (36) to reach

$$\tau^k = 1 - \frac{\beta^{-1} q^n(\rho;\kappa) - [\chi\rho + (1-\chi) q^n(\rho;\kappa)]\,(1-\delta)}{(\chi\rho + 1 - \chi)\, r(\rho)} \tag{37}$$

where $r(\rho) = [1 + \kappa(1 + \theta(\rho))]\,[\beta^{-1} - (1-\delta)] + \kappa\,[1 + \theta(\rho)]\,\chi(1-\delta)$. Therefore, there is a unique long-run level of τ^k, which could be either positive or negative. When $\kappa = 0$, we know from (14) and (18) that $q^n = 1$, $q^i = 1$, and thus $\rho = 1$. Condition (37) then implies that $\tau^k = 0$. $\square$

As discussed before, when $\kappa = 0$, the model is as if a standard neoclassical growth model. Not surprisingly, capital tax should be zero in the long run as found in previous studies such as Judd (1985), Chamley (1986), and Chari and Kehoe (1999). Additionally, when $\kappa = 0$, we know that $\rho = 1$ and that there is no reason for redistribution: a positive capital tax can only distorts capital accumulation, or the intertemporal marginal rates of substitutions. Therefore, a positive capital tax should be avoided.[6]

When $\kappa > 0$, the proposition again illustrates how complicate the issue of capital tax (or capital subsidy) is. There are three reasons for using τ^k, reflected by the three terms in $\Gamma(\rho)$.

First, taxes can increase with ρ as reflected by $(\chi\rho + 1 - \chi)$ in the denominator. When ρ rises, the planner wants less risk-sharing. There is a benefit of it because it raises the liquidity premium and push down real interest rate. Therefore, the government financing becomes less costly. Taxing capital can also drive buyers to demand more government bonds and can therefore drive down interest rate.

To see this channel, notice that when ρ rises, the constrained entrepreneurs are even more constrained. Workers today, expecting tomorrow's even tougher financing conditions, have a strong incentive to hold liquid government bonds, driving down the real interest rate. Alternatively, one can again use the Euler equation for bonds (23). We know that in the steady state, the real interest rate $P_{-1}R/P$ must satisfy

$$\frac{P_{-1}R}{P} = \beta^{-1}\,[\chi\rho + 1 - \chi]^{-1}$$

[6]When capital tax is not used, one can impose consumption tax. Zero capital tax in the long run implies constant consumption taxes in the long run.

In other words, the higher the ρ (the higher consumption inequality due to liquidity risks), the lower should the real interest rate be.

Second, taxes can decreases with ρ as reflected by $r(\rho)$,a decreasing function of ρ. When $\rho > 1$, we know that the planner cares about redistribution. This redistribution concern suggests that τ^k should fall with ρ. This is due to the externality arising from precautionary savings. Recall that when individual agents hold public liquidity, they do not internalize the effect on others. Namely, workers who have strong motives for precautionary savings drive down the interest rate; for other workers who also save in government bonds for precautionary reasons, they will have to save even more to prepare enough liquidity for future investment. To correct the externality, the government can step in by taxing capital less, or even subsidize capital, in order to avoid over-saving in government bonds.

Third, the government also values capital accumulation by private agents, which is reflected by the numerator $\beta^{-1}q^n(\rho;\kappa) - [\chi\rho + (1-\chi)q^n(\rho;\kappa)](1-\delta)$. This term can be positive or negative, meaning that the planner may prefer a capital tax or a capital subsidy. It can also increase or decrease with ρ. Intuitively, this term comes from the Euler equation of private agents, so that capital accumulation interacts with liquidity premium.

We shall see later numerical examples to illustrate the effect of ρ on planner's choice of capital tax (or subsidy). Still, one can summarize that there is an optimal level of liquidity premium (reflected by ρ) that the planner wants to target. The following discussion shows how one can solve the optimal level of liquidity premium, and how one can back out the unique long-run level of government debt to achieve this optimal level of liquidity premium.

4.3 The Debt-to-GDP Ratio in the Long Run

Finally, I analyze the long-run level of debt. To make the problem more interesting, I target a long-run government expenditure share of GDP G/Y (for example, 22% as in the post-war US experience). We need to solve for the steady-state level of C, L, ρ, and K, which depends on the initial condition K_{-1}. It turns out that with the class of utility functions (32), the long-run level of debt-to-GDP is independent of K_{-1}.

To see why the long-run debt-to-GDP ratio is uniquely determined, I start with a particular Φ that corresponds to a particular K_{-1}. To reach the debt-to-GDP ratio, it turns out that knowing ρ is enough. The following proposition proves the claim.

Proposition 7. *Suppose the government targets a long-run expenditure-to-GDP ratio G/Y, where $Y = AF(K,N) = rK/\alpha$. Then, risk-sharing ρ, capital tax τ^k, and debt-to-GDP ratio are all uniquely determined, and they are independent of the initial condition K_{-1}.*

Proof. Using J_C and J_ρ in (33) and (34), one has

$$\frac{J_\rho}{J_C} = \frac{\chi(1-\chi)\left[\rho^{-1}-1\right]\rho^{1/\sigma-1}}{\sigma\left[\chi+(1-\chi)\rho^{1/\sigma}\right]\left[\chi+(1-\chi)\rho^{1/\sigma-1}\right]}C \tag{38}$$

Notice that $J_\rho/J_C = \kappa\left[1-(1-\delta)(1-\chi)\right]\frac{\partial\theta}{\partial\rho}K$ from the planner's first-order condition (27). Therefore, aggregate consumption is proportional to the steady-state capital K

$$C = \frac{\kappa\sigma\left[1-(1-\delta)(1-\chi)\right]\left[\chi+(1-\chi)\rho^{1/\sigma}\right]\left[\chi\rho^{1-1/\sigma}+(1-\chi)\right]\rho}{\chi(1-\chi)\left[1-\rho\right]}\frac{\partial\theta}{\partial\rho}K$$

and C/K only depends on ρ.

Next, since $r = r(\rho) = \left[1+\kappa(1+\theta(\rho))\right]\left[\beta^{-1}-(1-\delta)\right]+\kappa\left[1+\theta(\rho)\right]\chi(1-\delta)$ from (35), I can write the feasibility condition (17) in a steady state as

$$C + G + K = r(\rho)K/\alpha + (1-\delta)K - \kappa\left[1+\theta(\rho)\right]\left[1-(1-\chi)(1-\delta)\right]K \tag{39}$$

One can see that since $G/Y = \alpha G/rK$ is a constant, we can divide $r(\rho)K$ on both sides and have one equation with only unknown ρ. Therefore, ρ is uniquely determined (note: if there are multiple solutions of ρ to the feasibility condition, the planner will pick the one that maximizes the social welfare. This also implies that τ^k is determined by (37)).

Finally, I will show that the debt-to-GDP ratio is also determined. I use the investment equation (10) and aggregate investment $X = \delta K$ to derive

$$\frac{Rb}{P} = c^i + \left[\frac{\delta\left[1-\phi(\rho)q^i(\rho)\right]}{\chi} - r(\rho) - (1-\delta)\phi(\rho)q^i(\rho)\right]K \tag{40}$$

By using the household's risk-sharing condition (20) $c^i = \frac{C}{\chi+(1-\chi)\rho_t^{1/\sigma}}$, we know that

$$c^i = \frac{\kappa\sigma\left[1-(1-\delta)(1-\chi)\right]\left[\chi\rho^{1-1/\sigma}+(1-\chi)\right]\rho}{\chi(1-\chi)\left(1-\rho\right)}\frac{\partial\theta}{\partial\rho}K$$

from (38). Now, the debt-to-GDP ratio in (40) can be expressed as

$$\begin{aligned}\frac{RB}{PY} = \frac{RB/P}{r(\rho)K/\alpha} &= \frac{\alpha\kappa\sigma\left[1-(1-\delta)(1-\chi)\right]\left[\chi\rho^{1-1/\sigma}+(1-\chi)\right]\rho}{(1-\chi)\left(1-\rho\right)r(\rho)}\frac{\partial\theta}{\partial\rho} \\ &+ \alpha\left[\frac{\frac{\delta\left[1-\phi(\rho)q^i(\rho)\right]}{\chi} - (1-\delta)\phi(\rho)q^i(\rho)}{r(\rho)} - 1\right]\end{aligned} \tag{41}$$

where I have used the equilibrium condition $b = B$. Therefore, the debt-to-GDP ratio only depends on ρ. Since ρ is uniquely determined, so is the debt-to-GDP ratio. $\square$

The long-run debt-to-GDP ratio deserves more discussions.

First, we know the capital/labor ratio K/L also depends only on $r(\rho)$ and therefore is uniquely determined. However, the level of labor hours depends on Φ (and thus depends on the initial level of capital stock K_{-1}). This is because Φ appears in the planner's marginal value of having one unit of leisure $J_{L,t}$:

$$J_{L,t} = (1+\Phi)v_l\left(\frac{L_t}{1-\chi}\right) + \Phi v_{ll}\left(\frac{L_t}{1-\chi}\right)\frac{L_t^2}{1-\chi} \tag{42}$$

That is, Φ still affects $J_{L,t}$. The above discussion implies that, although the capital/labor ratio is unique, both the long-run level of labor hours and the capital stock depend on the initial state of capital K_{-1}.

Second, fixing a particular ρ, we know that a higher σ raises the debt-to-GDP ratio in (41). To see this, when σ rises, the elasticity of intertemporal substitution falls. This means that the household is less sensitive to price movements, in particular less sensitive to the changes of liquidity premium.

For the government, it can use this response by increasing liquidity premium and reducing real interest rate. Then, the financing of government expenditures become cheaper, so that more debt issuance should be seen. Of course, when we change σ, the equilibrium ρ also changes. I leave the effect to the numerical example section.

Finally, what if the planner prefers a higher ρ, or less risk-sharing? Since $\partial\theta/\partial\rho < 0$, we know that a higher ρ implies a lower θ. As a result, asset price q^i, saleability ϕ, and rental rate of capital $r(\rho)$ fall. Using these co-movement in (41), we know that whether the debt-to-GDP ratio increases with ρ depends on

$$\frac{\partial \frac{\partial\theta/\partial\rho}{1-\rho}}{\partial\rho} = \frac{\frac{\partial^2\theta}{\partial\rho^2}(1-\rho) + \frac{\partial\theta}{\partial\rho}}{(1-\rho)^2}$$

Therefore, when $\frac{\partial^2\theta}{\partial\rho^2}(1-\rho) + \frac{\partial\theta}{\partial\rho} > 0$, it is sufficient to generate a higher debt-to-GDP ratio with a rising ρ. Since $\partial\theta/\partial\rho < 0$ and $\rho > 1$, it implies that $\frac{\partial^2\theta}{\partial\rho^2} < -\frac{\partial\theta}{\partial\rho}/(1-\rho)$, i.e., $\partial^2\theta/\partial\rho^2 < 0$ and it needs to be small enough. That is, $\theta(\rho)$ needs to be a decreasing concave function, and θ is not sensitive to the movement in ρ.

The economics behind this is straightforward. Issuing government debt partly drive away demand from the financial market. If $\partial^2\theta/\partial\rho^2 < 0$, then increasing ρ (or increasing liquidity premium, or reducing real interest rate) does not lead to large falls in financial market participation summarized by θ. The government should thus take advantage of this by issuing more debt, as the cost imposed on the financial market activity is small.

4.4 Numerical Examples

This section provides numerical examples to illustrate how one could relate the optimal long-run debt-to-GDP ratio in the model to the practice.

Almost all advanced economies exhibit balanced-growth paths, while $\sigma = 1$ is the only one that induces balanced growth path in the model with separable utility functions. Then, one might conjecture that for utility functions that are in some sense close to these utility functions, targeting debt-to-GDP ratio near and close to the one with $\sigma = 1$ should be a good first order approximation for the reality. Therefore, I continue with the class of utility (32), by setting $\sigma = 1$. Since $v(l)$ does not affect the optimal ρ , it does not affect the optimal debt-to-GDP ratio either. I set $v(l) = \mu l$ for simplicity.

The model is calibrated to annual data. The depreciation rate δ and the discount factor β are set to reasonable numbers 0.1 and 0.92. Capital share $\alpha = 0.33$ targets 16% investment-to-GDP ratio. Investment includes physical investment and Intermediation search costs. A fraction 24% of household members are entrepreneurs ($\chi = 0.24$) and have investment projects every year, which is the number to match investment spikes observed from U.S. manufacturing plants in Doms and Dunne (1998). I set the steady-state government expenditures share of GDP as $g = 22\%$, which is in line with the US post-war sample. I choose $\mu = 6.16$ such that the hours worked are 25% of total hours, but as explained it is not important for calculating the optimal level of debt.

In the following, I calibrate the financial market parameters ξ, η, and κ. There is no direct evidence on η, so I set $\eta = 1/2$. I do not use the implied optimal Ramsey allocation to calibrate ξ and κ, since the reality might not be optimal. Instead, I target return from government bonds, ignoring the policy that could generate it. I therefore use $\tilde{R} = 1.018$ to target annualized real (net) return of government bonds is 1.8%. This gives rise to $\rho = 1 + \left(\frac{1}{\beta R} - 1\right)/\chi = 1.2822.$[7]

To calibrate ξ and κ, I find two targets. I follow Del Negro, Eggertsson, Ferrero, and Kiyotaki (2017) targeting $\phi = 0.57$ which seems to match the average turnover of non government-issued assets in the flow-of-funds data. I also target a low $q^i = 1.05 > 1$ and entrepreneurs are indeed financing constrained. The exercise produces $\xi = 0.2942$, $\kappa = 0.0177$. With $\rho = 1.2822$, we can back out $\theta = 3.7540$.

Now, given parameters, I look at the optimal debt-to-GDP ratio, capital tax, and real interest rate. To solve the steady state, I solve ρ first. As the government targets expenditure-

[7] An entrepreneur consumes about 22% less than a worker in order to invest.

Table 2: Optimal Long-run Debt-to-GDP Ratios

	$\sigma = 1$	$\sigma = 2$	$\sigma = 5$	$\sigma = 10$
Debt-to-GDP	66.15%	73.37%	79.99%	83.58%
Capital Tax	-10.43%	-12.28%	-15.48%	-18.56%
Real Interest Rate	3.33%	2.76%	1.71%	0.66%
Liquidity Premium	5.37%	5.94%	6.99%	8.04%

GDP ratio at $g = 22\%$, (39) can be rewritten as:

$$\frac{\kappa\sigma\left[1-(1-\delta)(1-\chi)\right]\left[\chi+(1-\chi)\rho^{1/\sigma}\right]\left[\chi\rho^{1-1/\sigma}+(1-\chi)\right]\rho}{\chi(1-\chi)\left[1-\rho\right]}\frac{\partial\theta}{\partial\rho}$$
$$=\frac{r(\rho)}{\alpha}(1-g)-\delta-\kappa\left[1+\theta(\rho)\right]\left[1-(1-\chi)(1-\delta)\right]$$

One therefore solves ρ from the above equation (if there are multiple ρ, the planner picks the one that maximizes the social welfare). After obtaining ρ, we can use (41) to calculate the debt-to-GDP ratio. Capital tax rate can be computed from (37). Table (2) shows the results with different σ. It is widely accepted that $\sigma > 10$ is unreasonable, so it stops at 10.

When $\sigma = 1$, the optimal debt-to-GDP ratio is 66.15%, which is similar to what the US flow-of-funds point before 2008. Another interpretation is that when countries integrate their financial markets similar to the one calibrated, 66% is the long-run target for all countries if expenditures-to-GDP ratio is set to 22%.

When σ goes up, the elasticity of intertemporal substitutions $1/\sigma$ goes down. One finds that the need for public liquidity goes up since private agents prefer even more to smooth individual consumption over time, i.e., they are less sensitive to the rise of liquidity premium. The government can raise the liquidity premium and depress interest rate without much effect on the household. To do so, the government should issue more debt, and that is why the debt-to-GDP ratio rises with σ.

Although debt-to-GDP increases with σ, the level never exceeds 85%. Even 85% is less than most numbers observed in advanced economies nowadays. One can argue that changes in the financial market condition could lead to temporary increase or fall in government debt. In response to this, the numerical exercise suggests that the cost of providing public liquidity should be an important concern and permanently rolling-over debt at a level that is larger than 100% should be avoided.

Finally, capital tax is negative no matter what σ is, which means that the optimal policy is always to subsidize holding capital while taxing labor income. Liquidity premium rises faster and real interest rate falls faster, than the increase of debt-to-GDP ratio when σ

increases from 1 to 10. Notice that the rise of capital subsidy is also faster than the rise of debt-to-GDP ratio. Capital subsidy is thus mainly used for redistribution purposes to prevent ρ increases too fast, although obviously it encourages accumulation of real physical capital stock at the same time.

Notice that subsidizing capital is not a theoretical result, but a quantitative result. One can easily obtain positive capital tax for certain parameter values (for example, by setting $\xi = 0.4674$, $\kappa = 0.0446$). Given the parameters we have, capital subsidy rises with σ and the level of government debt. When one consider government subsidy as a particular expenditure, it should rise when the cost of financing the expenditure (i.e., real interest rate) falls.

5 Final Remark

In summary, search frictions imply that public liquidity competes with private liquidity. That is, government bonds can naturally provide liquidity service when private claims are subject to search frictions. The degree of liquidity service depends on the interactions between search frictions and government policies, and it can affect the liquidity constraints and risk sharing endogenously.

The key trade-off for the government is efficiency and inequality. The government wants to provide some liquidity service to reduce liquidity risks, and at the same time take advantage of the low interest rate arising from precautionary saving. Nevertheless, liquidity provision is costly due to distortionary taxation and the need to financing government expenditures.

As a result, both a long-run optimal level of government debt and a long-run real interest rate can exist. They can be also independent of the level initial capital stock. The calibration exercise suggests that for reasonable parameters, 65%-85% debt-to-GDP ratio is optimal.

A few related challenging questions remain. Should the government accumulate private assets by issuing government debt (e.g., quantitative easing)? If so, by how much? Accumulating private assets by issuing government bonds can finance government expenditures and provide liquidity. However, it will at the same time crowd out private investment, since liquidity frictions here are endogenous. One should add another policy choice variable (the fraction of capital owned by the government) in the Ramsey problem, and this policy choice is also an allocation variable.

What if the nominal interest rate is bounded below by the zero lower bound? The last question becomes especially important when nominal prices are sticky and the economy is subject to uncertainty. In addition, tax policies at the bound enter the center stage as it might have a large stimulating effect.

These questions are related to the current issues faced by many monetary and fiscal policies. I hope this paper provides a useful framework and solution method as a starting point.

References

Aiyagari, S. R., A. Marcet, T. J. Sargent, and J. Seppälä (2002): "Optimal taxation without state-contingent debt," *Journal of Political Economy*, 110(6), 1220–1254.

Aiyagari, S. R., and E. R. McGrattan (1998): "The Optimum Quantity of Debt," *Journal of Monetary Economics*, 42(3), 447–469.

Ajello, A. (2016): "Financial Intermediation, Investment Dynamics, and Business Cycle Fluctuations," *American Economic Review*, 106(8), 2256–2303.

Angeletos, G.-M., F. Collard, H. Dellas, and B. Diba (2013): "Optimal Public Debt Management and Liquidity Provision," NBER Working Paper 18800.

Barro, R. J. (1979): "On the Determination of the Public debt," *Journal of Political Economy*, pp. 940–971.

Bigio, S. (2012): "Liquidity Shocks and the Business Cycle," working paper, Columbia University.

Cao, M., and S. Shi (2014): "Endogenous Procyclical Liquidity, Capital Reallocation, and q," manuscript, Pennsylvania State University.

Chamley, C. (1986): "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," *Econometrica*, 54(3), 607–622.

Chari, V. V., L. J. Christiano, and P. J. Kehoe (1994): "Optimal Fiscal Policy in a Business Cycle Model," *Journal of Political Economy*, 102(4), 617–652.

Chari, V. V., and P. J. Kehoe (1999): "Optimal fiscal and monetary policy," *Handbook of macroeconomics*, 1, 1671–1745.

Cui, W. (2016): "Monetary-fiscal Interactions with Endogenous Liquidity Frictions," *European Economic Review*, 87, 1–25.

Cui, W., and S. Radde (2015): "Search-Based Asset Liquidity and the Macroeconomy," working paper, University College London.

Cui, W., and S. Radde (2016): "Money and Asset Liquidity in Frictional Capital Markets," *American Economic Review*, 106(May Issue), 496–502.

Del Negro, M., G. Eggertsson, A. Ferrero, and N. Kiyotaki (2017): "The Great Escape? A Quantitative Evaluation of the Fed's Liquidity Facilities," *American Economic Review*, 107(3), 824–857.

Doms, M., and T. Dunne (1998): "Capital Adjustment Patters in Manufacturing Plants," *Review of Economic Dynamics*, 1, 409–429.

Gertler, M., and P. Karadi (2011): "A model of unconventional monetary policy," *Journal of Monetary Economics*, 58(1), 17–34.

Gertler, M., and N. Kiyotaki (2010): "Financial Intermediation and Credit Policy in Business Cycle Analysis," *Handbook of Monetary Economics*, 3(3), 547–599, working paper, NYU and Princeton University.

Holmström, B., and J. Tirole (1998): "Private and Public Supply of Liquidity," *Journal of Political Economy*, 106(1), 1–40.

Judd, K. L. (1985): "Redistributive Taxation in A Simple Perfect Foresight Model," *Journal of Public Economics*, 28(1), 59–83.

Kiyotaki, N., and J. Moore (2012): "Liquidity, Business Cycles, and Monetary Policy," NBER Working Paper 17934.

Lagos, R., and G. Rocheteau (2008): "Money and Capital as Competing Media of Exchange," *Journal of Economic Theory*, 142(1), 247–258.

Lucas, R. J., and N. L. Stokey (1983): "Optimal Fiscal and Monetary Policy in an Economy without Capital," *Journal of Monetary Economics*, 12(1), 55–93.

Nezafat, M., and C. Slavik (2010): "Asset Prices and Business Cycles with Financial Shocks," Social Science and Research Network working paper 1571754.

Rocheteau, G. (2011): "Payments and Liquidity under Adverse Selection," *Journal of Monetary Economics*, 58(3), 191–205.

Rocheteau, G., and P.-O. Weill (2011): "Liquidity in Frictional Asset Markets," *Journal of Money, Credit and Banking*, 43(s2), 261–282.

Shi, S. (2015): "Liquidity, Assets and Business Cycles," *Journal of Monetary Economics*, 70, 116–132.

Vayanos, D., and J. Wang (2013): "Market Liquidity: Theory and Empirical Evidence," in *Handbook of the Economics of Finance*, ed. by G. Constantinides, M. Harris, and R. Stulz, chap. 19.

Woodford, M. (1990): "Public Debt as Private Liquidity," *American Economic Review*, 80(2), 382–388.

Zhu, X. (1992): "Optimal Fiscal Policy in a Stochastic Growth Model," *Journal of Economic Theory*, 58(2), 250–289.

Appendix

Proof of Proposition 1

Since $\frac{\kappa}{f} = q - q^i$, then $1 - \phi q^i = \frac{\kappa\phi}{f} + 1 - \phi q$ and

$$q^r = \frac{1 - \phi q^i}{1 - \phi} = \frac{\frac{\kappa\phi}{f} + 1 - \phi q}{1 - \phi}$$

To minimize q^r, I derive the first-order condition with respect to ϕ:

$$\frac{dq^r(\phi)}{d\phi} = \frac{(1-\phi)\left[\frac{\kappa}{(1-\eta)f} - q\right] + \left[\frac{\kappa\phi}{f} + 1 - \phi q^n\right]}{(1-\phi)^2} = 0 \tag{43}$$

where I have used the fact that $f = \xi^{\frac{1}{1-\eta}} \phi^{\frac{\eta}{\eta-1}}$. Rearranging, I obtain

$$q = 1 + \kappa\left[\frac{1-\phi}{(1-\eta)f} + \frac{\phi}{f}\right] \tag{44}$$

which implies that

$$q^i = q - \frac{\kappa}{f} = 1 + \frac{\kappa\eta(1-\phi)}{(1-\eta)f}$$

as shown in the main text. I also check the second-order condition to ensure minimization (available upon request). Finally, one can directly compute

$$\frac{\partial q^i}{\partial \theta} = \frac{\kappa\eta}{1-\eta} \frac{\eta\theta^{\eta-1} - \xi}{\xi}$$

Therefore, $\frac{\partial q^i}{\partial \theta} > 0$ if and only if $\theta < \left(\frac{\eta}{\xi}\right)^{\frac{1}{1-\eta}}$.

Proof of Proposition 2

When private claims and government bonds co-exist, I can use the two first-order conditions 21 and 23 to derive

$$\chi\rho_{t+1} r^{ni}_{t+1} + (1-\chi) r^{nn}_{t+1} = \chi\rho_{t+1} P_t R_t / P_{t+1} + (1-\chi) P_t R_t / P_{t+1}$$

Rearrange one has

$$\chi\rho_{t+1}\left(r^{ni}_{t+1} - P_t R_t / P_{t+1}\right) + (1-\chi)\left(r^{nn}_{t+1} - P_t R_t / P_{t+1}\right) = 0$$

Notice that that r^{ni}_{t+1} and r^{nn}_{t+1} cannot be both larger than $P_t R_t / P_{t+1}$. Otherwise, no one holds government bonds because the return is strictly dominated by private claims. Therefore, given that $r^{ni}_{t+1} < r^{nn}_{t+1}$ (because $q^n_{t+1} > 1$ when $\kappa > 0$) we thus know

$$r^{ni}_{t+1} < P_t R_t / P_{t+1} < r^{nn}_{t+1}$$

Then, the liquidity premium is positive since

$$\begin{aligned}\Delta &= \chi r^{ni}_{t+1} + (1-\chi) r^{nn} - P_t R_t / P_{t+1} = \chi \left(r^{ni}_{t+1} - P_t R_t / P_{t+1} \right) + (1-\chi) \left(r^{nn}_{t+1} - P_t R_t / P_{t+1} \right) \\ &> \chi \rho_{t+1} \left(r^{ni}_{t+1} - P_t R_t / P_{t+1} \right) + (1-\chi) \left(r^{nn}_{t+1} - P_t R_t / P_{t+1} \right) \\ &= 0\end{aligned}$$

where I have used the fact that $\rho_{t+1} > 1$.

Proof of Proposition 3

Since we know that

$$q^i_t = 1 + \frac{\kappa_t \eta \left(1 - \phi_t\right)}{\left(1 - \eta\right) f_t}$$

from Proposition 1 and $q^n_t = q^i_t + \kappa_t \left(\frac{1}{\phi_t} + \frac{1}{f_t} \right)$, I can use the definition to derive

$$\begin{aligned}\rho &= \frac{q^n}{q^r} = \frac{(1-\phi) q^n}{1 - \phi q^i} = \frac{1 + \frac{\kappa\eta}{1-\eta}\frac{1-\phi}{f} + \frac{\kappa}{f} + \frac{\kappa}{\phi}}{1 - \frac{\kappa\eta}{1-\eta}\frac{1}{f}} \\ &= 1 + \frac{\kappa \left[\frac{1}{(1-\eta) f} + \frac{1}{\phi} \right]}{1 - \frac{\kappa\eta}{1-\eta}\theta} \\ &= 1 + \frac{\theta + (1-\eta)}{\xi \theta^{1-\eta} \left[(1-\eta)\kappa^{-1} - \eta\theta \right]}\end{aligned}$$

where $f(\theta) = \xi \theta^{-\eta}$ and time subscripts are omitted to save notation. Notice that $0 < \theta < \frac{1-\eta}{\kappa\eta}$ since $\rho > 1$. Using the implicit function theorem, one knows that

$$\frac{\partial \theta}{\partial \rho} = \frac{\xi \theta^{1-\eta} \left[(1-\eta)\, \kappa^{-1} - \eta\theta \right]}{1 - (\rho - 1) \xi \theta^{-\eta} \left[\kappa^{-1} (1-\eta)^2 - \eta (2-\eta) \theta \right]}$$

Then, plugging $\rho = 1 + \frac{\theta + (1-\eta)}{\xi\theta^{1-\eta}[(1-\eta)\kappa^{-1} - \eta\theta_t]}$ again, we know that

$$\frac{\partial\theta}{\partial\rho} = \frac{\xi\theta^{1-\eta}\left[(1-\eta)\kappa^{-1} - \eta\theta\right] - \left[\theta + (1-\eta)\right]\left[(1-\eta)^2\kappa^{-1}\xi\theta^{-\eta} - \xi\eta(2-\eta)\theta^{1-\eta}\right]}{\left[\xi\theta^{1-\eta}\left[(1-\eta)/\kappa - \eta\theta\right]\right]^2}$$

From the above expression $\partial\theta/\partial\rho < 0$ if and only if the numerator is negative, which is equivalent to

$$\theta^2 + \left(\kappa^{-1} + 2 - \eta\right)\theta - \frac{(1-\eta)^2}{\kappa\eta} < 0$$

after some algebra. Therefore, $\partial\theta/\partial\rho < 0$ if and only if

$$0 < \theta < \frac{-\left(\kappa^{-1} + 2 - \eta\right) + \sqrt{\left(\kappa^{-1} + 2 - \eta\right)^2 + \frac{4(1-\eta)^2}{\kappa\eta}}}{2}$$

and $\partial\theta/\partial\rho > 0$ if and only if

$$\frac{-\left(\kappa^{-1} + 2 - \eta\right) + \sqrt{\left(\kappa^{-1} + 2 - \eta\right)^2 + \frac{4(1-\eta)^2}{\kappa\eta}}}{2} < \theta < \frac{1-\eta}{\kappa\eta}.$$

Proof of Proposition 4

As in the definition, any competitive equilibrium allocation needs to satisfy the feasibility condition. One only needs to prove the implementability condition. First, multiply a worker's budget constraint (6) by $(1-\chi)$ and an entrepreneur's budget constraint (8) by $\chi\rho_t$:

$$\chi\rho_t g\left(\rho_t u_{ct}^n\right) + (1-\chi)c_t^n + q_t^n K_t + \frac{B_t}{P_t} = (1-\tau_t^l)w_t L_t + \rho_t^K K_{t-1} + \rho_t^\chi \frac{R_{t-1}B_{t-1}}{P_t} \tag{45}$$

where I have directly used $c_t^i = g\left(\rho_t u_{ct}^n\right)$ and

$$\rho_t^K \equiv \rho_t^\chi r_t(1-\tau_t^k) + \left[\chi\rho_t + (1-\chi)\, q_t^n\right](1-\delta)$$

$$\rho_t^\chi \equiv \chi\rho_t + 1 - \chi$$

Second, multiplying both sides of 45 by $\beta^t u(c_t^n)$, we get

$$\beta^t u_{c,t}^n\left[\chi\rho_t g\left(\rho_t u_{ct}^n\right) + (1-\chi)c_t^n + q_t^n K_t + \frac{B_t}{P_t}\right] = \beta^t u_{c,t}^n\left[(1-\tau_t^l)w_t L_t + \rho_t^K K_{t-1} + \frac{\rho_t^\chi R_{t-1}}{P_t}B_{t-1}\right] \tag{46}$$

Using the first-order conditions for capital, bonds, and labor (21), (19), and (23), we have

$$\beta^t u^n_{c,t} q^n_t = \beta^{t+1} u^n_{c,t} \rho^K_{t+1}, \quad \beta^t u^n_{c,t}/P_t = \beta^{t+1} u^n_{c,t} \rho^\chi_{t+1} R_t/P_{t+1}, \quad (1-\tau^l_t) w_t u^n_{c,t} = -u^n_{l,t}$$

and we can simplify (46) to

$$\begin{aligned}
&\beta^t \left[\chi \rho_t u^n_{c,t} g\left(\rho_t u^n_{c,t}\right) + (1-\chi) u^n_{c,t} c^n_t + u^n_{l,t} L_t\right] + \beta^{t+1} u^n_{c,t+1} \left(\rho^K_{t+1} K_t + \frac{\rho^B_{t+1} R_t}{P_{t+1}} B_t\right) \\
=&\beta^t u^n_{c,t} \left(\rho^K_t K_{t-1} + \frac{\rho^B_t R_{t-1}}{P_t} B_{t-1}\right).
\end{aligned}$$

Now, after summing the above constraints over all periods $t = 0, 1, 2, ..., T$ and letting $T \to \infty$, I apply the transversality conditions for financial assets,[8] and I retrieve the implementability condition in (24)

$$\sum_{t=0}^{\infty} \beta^t \left[u^n_{c,t} \left[\chi \rho_t g\left(\rho_t u^n_{c,t}\right) + (1-\chi) c^n_t\right] + u^n_{l,t} L_t\right] - H = 0$$

where H is given by

$$H = u^n_{c,0} \left(\rho^K_0 K_{t-1} + \frac{\rho^B_0 R_{-1}}{P_0} B_{-1}\right)$$

Finally, one should substitute out $R_{-1} B_{-1}/P_0$ in H. We know that from the flow-of-funds constraints of entrepreneurs at time 0

$$c^i_0 + \chi^{-1}(1-\phi_0 q^i_0)\left[K_0 - (1-\delta) K_{-1}\right] = r_0(1-\tau^k_0) K_{-1} + \phi_0 q^i_0 (1-\delta) K_{-1} + \frac{R_{-1}}{P_0} B_{-1}$$

Then one can solve for $R_{-1} B_{-1}/P_0$ and use $c^i_0 = g(\rho_0 u^n_{c,0})$ to reach (25).

[8]The transversality conditions (for capital and liquid assets) are the no-arbitrage conditions

$$\lim_{T\to\infty} \beta^T u^n_{c,T} \rho^K_T K_T = 0, \quad \lim_{T\to\infty} \beta^T u'(c^n_{c,T}) \rho^\chi_T R_T B_T / P_T = 0$$

Proof of Lemma 1

From the definition of J, we have $J_{C,t}$ and $J_{\rho,t}$ as

$$\begin{aligned} J_{C,t} &= \chi u_c\left(h^i(C_t,\rho_t),0\right)h_c^i(C_t,\rho_t) + (1-\chi)u_c\left(h^n(C_t,\rho_t),\frac{L_t}{1-\chi}\right)h_c^n(C_t,\rho_t) \\ &+ \Phi u_{lc}\left(h^n(C_t,\rho_t),\frac{L_t}{1-\chi}\right)L_t h_c^n(C_t,\rho_t) \\ &+ \Phi u_{cc}\left(h^n(C_t,\rho_t),\frac{L_t}{1-\chi}\right)\left[\chi\rho_t h^i(C_t,\rho_t) + (1-\chi)h^n(C_t,\rho_t)\right]h_c^n(C_t,\rho_t) \\ &+ \Phi u_c\left(h^n(C_t,\rho_t),\frac{L_t}{1-\chi}\right)\left[\chi\rho_t h_c^i(C_t,\rho_t) + (1-\chi)h_c^n(C_t,\rho_t)\right] \end{aligned} \tag{47}$$

$$\begin{aligned} J_{\rho,t} &= \chi u_c\left(h^i(C_t,\rho_t),0\right)h_\rho^i(C_t,\rho_t) + (1-\chi)u_c\left(h^n(C_t,\rho_t),\frac{L_t}{1-\chi}\right)h_\rho^n(C_t,\rho_t) \\ &+ \Phi u_{lc}\left(h^n(C_t,\rho_t),\frac{L_t}{1-\chi}\right)L_t h_\rho^n(C_t,\rho_t) \\ &+ \Phi u_{cc}\left(h^n(C_t,\rho_t),\frac{L_t}{1-\chi}\right)\left[\chi\rho_t h^i(C_t,\rho_t) + (1-\chi)h^n(C_t,\rho_t)\right]h_\rho^n(C_t,\rho_t) \\ &+ \Phi u_c\left(h^n(C_t,\rho_t),\frac{L_t}{1-\chi}\right)\left[\chi h^i(C_t,\rho_t) + \chi\rho h_\rho^i(C_t,\rho_t) + (1-\chi)h_\rho^n(C_t,\rho_t)\right] \end{aligned}$$

One can see that when $u(c,l) = \frac{c^{1-\sigma}-1}{1-\sigma} - v(l)$, we have $(c_t^i)^{-\sigma} = \rho_t\,(c_t^n)^{-\sigma}$. Additionally, $\chi c_t^i + (1-\chi)c_t^n = C_t$ and therefore the following is true

$$c_t^i = h^i(C_t,\rho_t) = \frac{C_t}{\chi + (1-\chi)\rho_t^{1/\sigma}}$$

$$c_t^n = h^n(C_t,\rho_t) = \frac{\rho_t^{1/\sigma}C_t}{\chi + (1-\chi)\rho_t^{1/\sigma}}$$

$J_{C,t}$ can thus be simplified to

$$\begin{aligned} J_{C,t} &= \frac{\chi\rho_t + (1-\chi)\rho_t^{1/\sigma}}{\chi + (1-\chi)\rho_t^{1/\sigma}}\left[1+(1-\sigma)\Phi\right](c_t^n)^{-\sigma} \\ &= \frac{\chi + (1-\chi)\rho_t^{1/\sigma-1}}{\left[\chi + (1-\chi)\rho_t^{1/\sigma}\right]^{1-\sigma}}\left[1+(1-\sigma)\Phi\right]C_t^{-\sigma} \end{aligned}$$

$J_{\rho,t}$ can thus be simplified to

$$\begin{aligned} J_{\rho,t} &= \frac{\chi(1-\chi)\left[\rho_t^{-1}-1\right]}{\sigma\left[\chi+(1-\chi)\rho_t^{1/\sigma}\right]}\left[1+(1-\sigma)\Phi\right](c_t^n)^{1-\sigma} \\ &= \frac{\chi(1-\chi)\left[\rho_t^{-1}-1\right]\rho_t^{1/\sigma-1}}{\sigma\left[\chi+(1-\chi)\rho_t^{1/\sigma}\right]^{2-\sigma}}\left[1+(1-\sigma)\Phi\right]C_t^{1-\sigma} \end{aligned}$$

We therefore know that

$$\frac{J_{\rho,t}}{J_{C,t}} = \frac{\chi(1-\chi)\left[\rho_t^{-1}-1\right]\rho_t^{1/\sigma-1}}{\sigma\left[\chi+(1-\chi)\rho_t^{1/\sigma}\right]\left[\chi+(1-\chi)\rho_t^{1/\sigma-1}\right]}C_t.$$